# Nonlinear Dynamics of Solitary Wave Structures in Complex Continuous Media


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***Abstract*—The results of simulation of evolution and interaction of the multidimensional soliton structures being described by the equations of nonlinear BK system, namely: generalized KP equation, 3-DNLS and 3-GNLS equations are presented. It is shown that the collision interaction can be elastic and non-elastic with the *N*-soliton structures' formation. The results are in conformity with the stability conditions obtained earlier analytically for the equations of the BK system.**



***Keywords—solitons, BK system, generalized KP equation, DNLS equation, nonlinear Schrödinger equation, numerical simulation***


## I. Introduction. Basic Equations

The system describing the dynamics of a wide class of multidimensional nonlinear wave processes in complex continuous media, known as the Belashov-Karpman (BK) system, is written as [1]

$$\partial_t u + \hat{A}(t,u)u = f,\ \ f = \kappa\int_{-\infty}^{x}\Delta_\perp u dx,\ \ \Delta_\perp = \partial_y^2 + \partial_z^2 \quad (1)$$

and if the differential operator has form

$$\hat{A}(t,u) = \alpha u\partial_x - \partial_x^2(\nu - \beta\partial_x - \gamma\partial_x^3), \quad (2)$$

it is the 3-dimensional (3D) generalized Kadomtsev-Petviashvili (GKP) equation [1]-[3]

$$\partial_x(\partial_t u + \alpha u\partial_x u - \nu\partial_x^2 u + \beta\partial_x^3 u + \gamma\partial_x^5 u) = \kappa\Delta_\perp u, \quad (3)$$
$$\kappa = -c_0/2$$

which in dependence on physical sense of function $u$ and the coefficients describes the ion-acoustic and fast magnetosonic (FMS) waves in a plasma, nonlinear structures of the IGW type in the upper atmosphere (ionosphere), waves on surface of "shallow" water and etc. [1]-[4].

If the differential operator in (1) has form

$$\hat{A}(t,u) = 3s|p|^2 u^2\partial_x - \partial_x^2(i\lambda + \nu) \quad (4)$$

where

$$u = h = (B_y + iB_z)/2B_0\,|1-\beta|^{1/2},\ \ \mathbf{h} = \mathbf{B}_\perp/B_0,$$
$$p = (1+ie), \quad (5)$$

the BK system takes form of the 3D generalized derivative nonlinear Schrödinger (3-DNLS) equation:

$$\partial_t h + s\partial_x\left(|h|^2 h\right) - i\lambda\partial_x^2 h - \nu\partial_x^2 h = \sigma\int_{-\infty}^{x}\Delta_\perp h dx \quad (6)$$

which describes propagation of the Alfvén waves and solitons in the magnetized plasma [1]-[3].

In case when operator in (1) has form

$$\hat{A}(t,u) = i\,[\gamma|u|^2 - \beta\partial_x^2] + \alpha/2, \quad (7)$$

(1) turns to the 3D generalized nonlinear Schrödinger (3-GNLS) equation [5]

$$\partial_t u + i\gamma|u|^2 u - i\beta\partial_x^2 u + (\alpha/2)u = \sigma\int_{-\infty}^{x}\Delta_\perp u\, dx + f' \quad (8)$$

where $\alpha, \beta, \gamma = \varphi(t,x,y,z)$, $f' = f'(t,x,y,z)$, which describes the dynamics of the envelop of modulated nonlinear waves and pulses in weakly nonlinear media with dispersion, and has numerous important applications [5]: in plasma physics (propagation of the Langmuir waves in hot plasmas), nonlinear optics (propagation of light pulses in nonlinear optical fibers and planar waveguides) and hydrodynamics (propagation of the small-amplitude gravity waves on surface of deep inviscid fluid), and other.

As it was shown analytically in [1]-[3], [5], [6], (3), (6) and (8) at $\nu, \alpha=0$ can have stable 1D, 2D and 3D soliton solutions. This work is devoted to numerical study of dynamics of 2D, 3D and 1D (due to clarity) soliton structures in order to confirm the previously obtained by us analytical results.

## II. Modeling of the Dynamics of Soliton Structures

At modeling, the specially developed in [1]-[3] for the equations of the BK system highly accurate methods based on both finite difference and spectral approaches were used. In numerical experiments we studied the evolution and the collisional interactions of soliton structures described by (3), (6) and (8), when, according to theory, for $\nu, \alpha=0$ stable solutions should take place. We present below some of the main results.

In experiments using the GKP model we have found that solitons with smooth (algebraic) asymptotics, when $\nu=0$ and $\gamma > 0$, $\beta \le 0$ in (3), interact trivially exchanging their momenta and energy (Fig. 1) regardless of distance $\Delta r(0)$ between them.

However, in case of $\gamma$, $\beta > 0$, the dynamics of solitons having oscillating asymptotics in nontrivial and the result of the interaction depends on the ratio of the amplitudes of the initial pulses and $\Delta r$ (or $\Delta x$ when moving along the $x$ axis) at $t$=0. As one can see from the example presented in Fig. 2, in the case of significantly different amplitudes $u_1(0)$ and $u_2(0)$ at relatively small distance $\Delta x(0)$, the interaction of the pulses


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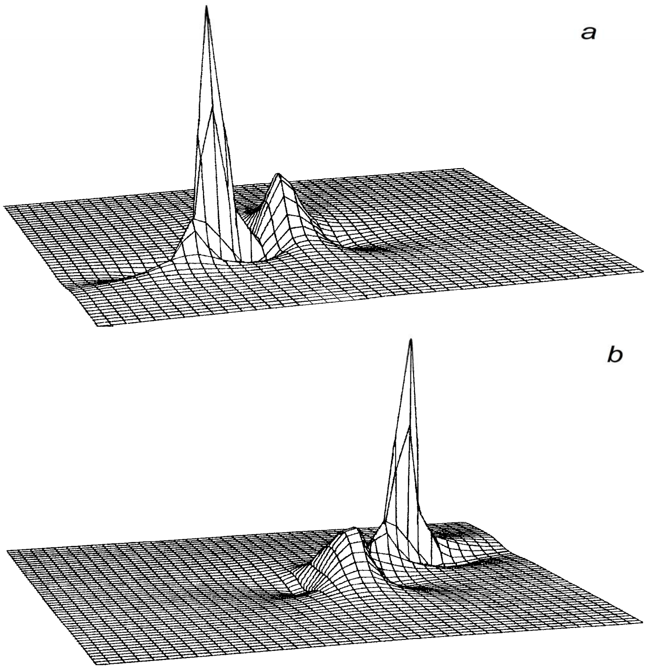


Fig. 1. Collision of the 2D solitons with algebraic asymptotics at $u_1(0)=12$, $u_2(0)=4$, $\Delta r(0)=3.3$: (a) $t=0$; (b) $t=0.7$.

has an inelastic character and, as a result, one pulse with the oscillating structure of the tails, which corresponds to the 2D GKP soliton, is formed.

Qualitatively similar results were obtained when solitons with amplitudes close each other interact at $\Delta x(0) \leq u_{1,2}(0)$: an oscillating structure is also formed at evolution and the pulse having a smaller amplitude, is "absorbed" by the tail of larger soliton, and one soliton with $u_2(0) < u < u_1(0)$ and oscillating asymptotics is formed.

The cases when $\Delta x(0)$ is larger than the characteristic sizes of the interacting pulses and the amplitudes are close, have particular interest. In these cases, the interaction leads to the formation of a bound state and a soliton structure with two maxima and oscillating tails, 2D bi-soliton of the KP equation, is formed. We note that the possibility of the existence of such structures was first noted in [7], but the dynamics of their formation and stability were first investigated in [2].

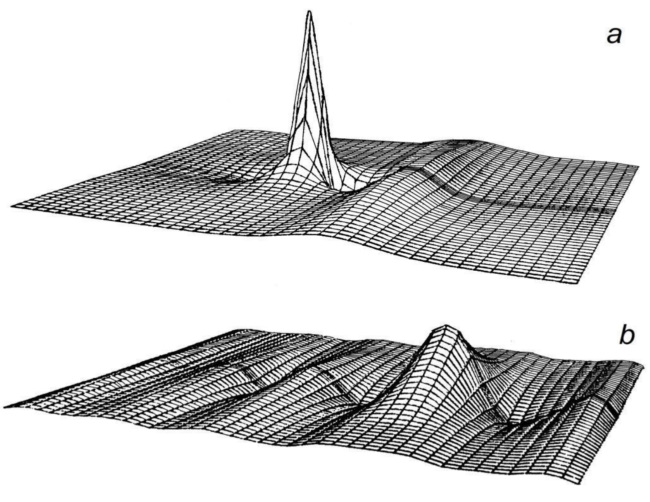


Fig. 2. Formation of the 2D soliton with oscillating structure at interaction initial pulses $u_1(0)=8$, $u_2(0)=1$ at $\Delta x(0)=4$: (a) $t=0$; (b) $t=0.8$.

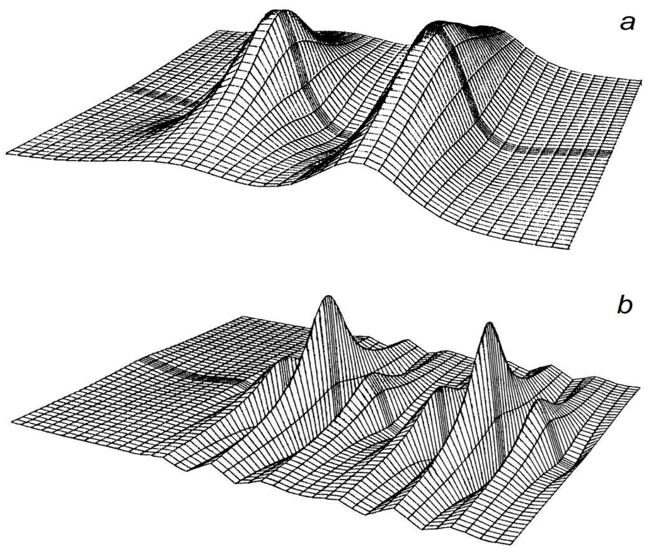


Fig. 3. Formation of the 2D bi-soliton of the GKP equation at $u_1(0)=1.35$, $u_2(0)=1.3$, $\Delta x(0)=6$: (a) $t=0$; (b) $t=0.9$.

Here we investigated the correspondence of numerical results to the analytical ones [2]. The results shown in Figs. 1-3 correspond to the cases when the characteristics of the propagation medium satisfy the conditions of solutions' stability.

In cases when $\nu > 0$ in the GKP equation (the dissipation effects are significant [2]), on a level with general damping of the wave field amplitude a change in the structure of 2D solitons is observed: the effect of lengthening the soliton "tail", a decrease of the oscillations' frequency and damping of oscillations behind the main maximum, as well as asymmetrical changes of the integrals corresponding to the momentum and energy in the frontal and rear "cavities" (where $u < 0$) are observed [1], [ 2].

The 3D problems in the framework of the GKP model are associated mainly with the study of the dynamics of the FMS waves' beams in a magnetized plasma. They were studied in detail in [1], [3], [4], [6] and we will not dwell on them here.

Let us consider the results obtained in modeling the evolution of the 3D soliton structures in the framework of the 3-DNLS model (6). Problem of stability of the 3-DNLS solutions was analytically investigated for the first time in [8] (see also [1], [6]), and here we confirm experimentally these results. In our numerical modeling we used the axially-symmetric geometry when $\Delta_\perp = \partial_\rho^2 + (1/\rho)\partial_\rho$, $\rho^2 = y^2 + z^2$. The initial conditions are taken in the form of the axially-symmetric solitary pulses of two types:

- soliton-like axially symmetric pulse:

$$h(x,\rho,0) = h_0(x)\exp\left[i\varphi(x) - \rho^2 / l_\rho^2\right] \quad (9)$$

with

$$h_0(x) = 2\sqrt{2}\delta\sin\vartheta\left[\cosh(4\delta^2\sin\vartheta x) + \cos\vartheta\right]^{-1/2} \quad (10)$$

and $\varphi(x) = -2s\delta^2\cos\vartheta x - (3s/4)\int_{-\infty}^{x} h_0^2(x)dx$ where $0 < \vartheta < \pi$;

- modulated plane wave:

$$h(x,\rho,0) = H_0 \exp\left(2\pi i x/\lambda - x^2/l_x^2 - \rho^2/l_\rho^2\right) \quad (11)$$

where $\lambda$ is the wavelength, $H_0$ is the amplitude, and $l_x$ and $l_\rho$ are the characteristic scales of the Gaussian envelop modulation in the $x$ and $\rho$-directions. Note that for $\rho = 0$, the initial conditions (9) and (11) are equivalent to those used for the numerical simulation of the evolution of the 1D Alfvén wave in [9].

Thus, for non-dissipative case, when $\nu=0$ in (6), we have obtained the following numerical results.

- For $\lambda=1$, $s = -1$, large $\sigma>0$, and the initial pulse weakly limited in the transverse $\rho$-direction when the stability condition [8] is satisfied, the evolution for large $t$ results in formation of the stable 3D (axially-symmetric) solution (Fig. 4).
- At the opposite signs of $\lambda$ and $s$ [that is equivalent to change $t \to -t$, $\sigma \to -\sigma$ in Eqs. (1), (3)] a 3D Alfvén wave spreads with evolution (Fig. 5). This completely corresponds our analytical results [8] on stability of the solutions of the 3-DNLS equation.
- At $\lambda=1$, $s = -1$ for small $\sigma>0$ and initial pulse rather strong limited in the $\rho$-direction, in the numerical experiments one can observe development of the 3D collapsing solutions of the 3-DNLS equation (Fig. 6). Note, that the same effects have been observed in the systems describing the evolution of the FMS waves [10] and Langmuir waves [11] waves in a plasma.

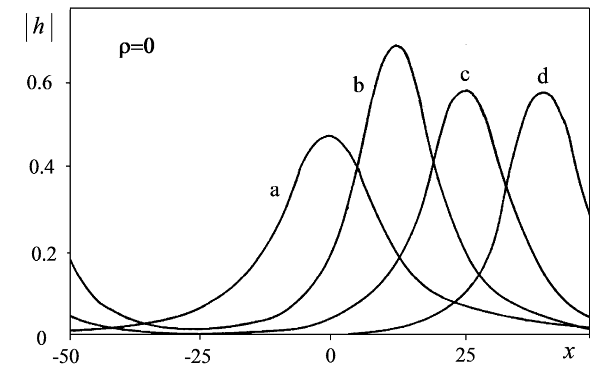


Fig. 4. Evolution of a 3D right circularly polarized nonlinear pulse (9) for $\lambda=1$, $s = -1$, $\sigma=1$: a) $t=0$, b) $t=25$, c) $t=50$, d) $t=75$.

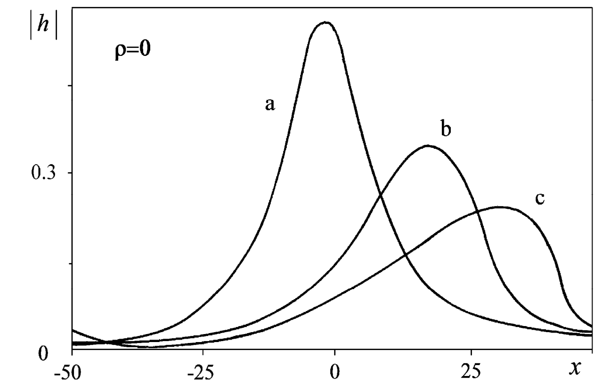


Fig. 5. Evolution of a 3D right circularly polarized nonlinear pulse (11) for $\lambda=-1$, $s = 1$, $\sigma=1$: a) $t=0$, b) $t=50$, c) $t=100$.

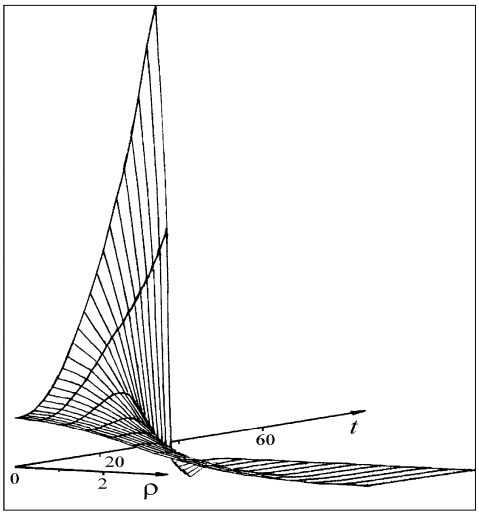


Fig. 6. Dynamics of a 3D right circularly polarized nonlinear pulse (11) (cross-section by the $\rho$ -plane in the point $h_{\max}$) for $\lambda=1$, $s = -1$, $\sigma = 0.2$.

Let us now consider the main results obtained by us in modeling the evolution and interaction of soliton structures in the framework of the GNLS model (8). Problem of stability of the NLS solutions was analytically investigated in [5], and the goal here was to experimentally verify these results when the medium is nonstationary and nonuniform.

Fig. 7 shows the result of simulation in the simplest 1D case ($\sigma = 0$) with $\beta(t) = -0.5(1+\sin 0.1\pi t)$, $\gamma = -1$ and negative nonlinearity when dissipation and external influences can be neglected ($\alpha$, $f$= 0). The condition of stability of the solution (see [5]) is satisfied. One can see that the evolution of the initial envelop pulse

$$u(x,0) = A\exp(-x^2/l), \quad A = \sqrt{|\beta/\gamma|} \quad (12)$$

is accompanied by its pulsations with a pulse shift in the direction of the $x$ axis. In case when $\beta = 0.5$, $\gamma = -1+0.01\sin 2\pi t$, more strong stable pulsations of breather type without any shift of the pulse are observed (Fig. 8). In both cases the regimes of so-called quasi-stable evolution are realized [5].

Fig. 9 shows the results of study of interaction of two GNLS pulses in dependence on initial distance between them at weak negative nonlinearity and no dissipation when stability condition is satisfied.

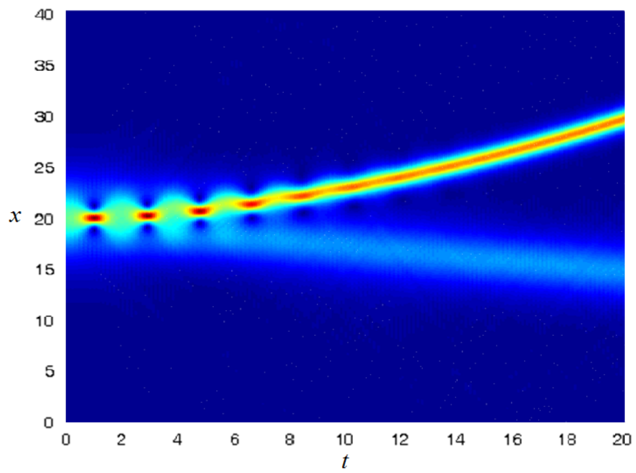


Fig. 7. Evolution of a 1D Gaussian envelop pulse in nonstationary medium at $\beta(t) = -0.5(1+\sin 0.1\pi t)$, $\gamma = -1$.

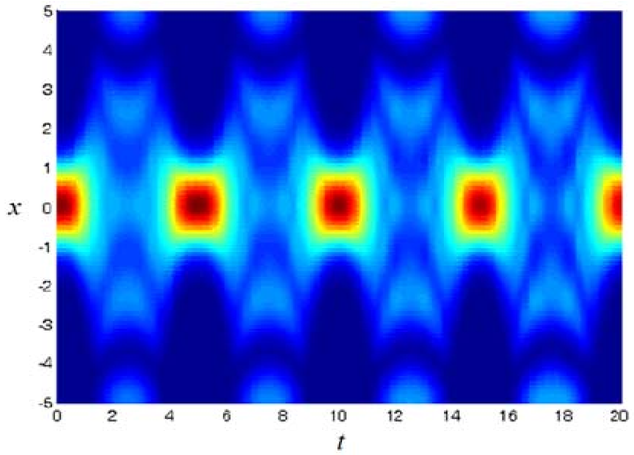


Fig. 8. Evolution of a 1D Gaussian envelop pulse in nonstationary medium at $\beta = 0.5$, $\gamma = -1+0.01\sin 2\pi t$.

One can see that the result depends on value of $s$ at $t = 0$, and with decreasing $s$ one can observe a transition from stable state to the regime of stable pulsation of the breather type.

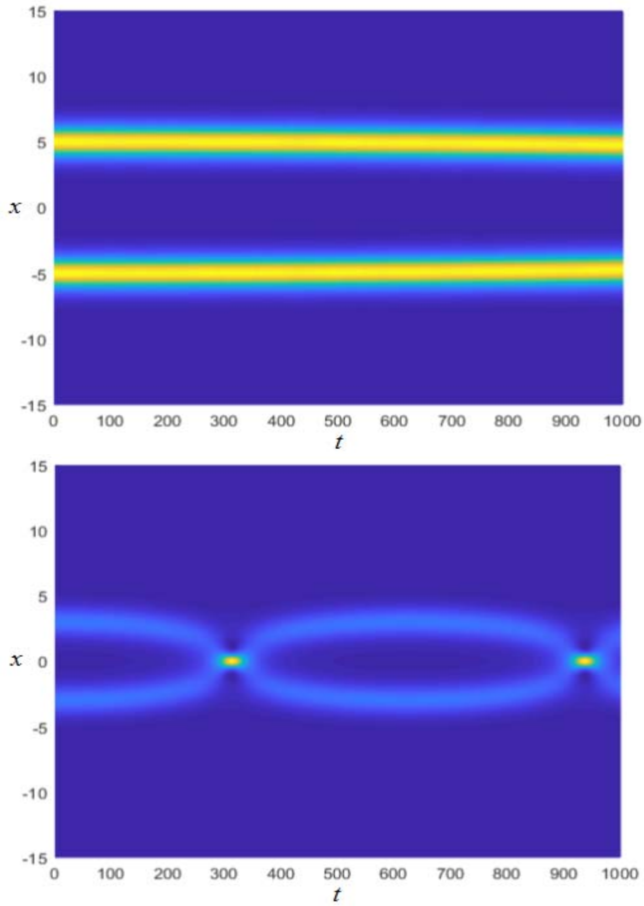


Fig. 9. Interaction of two pulses $u_0 = A[\mathrm{sch}(x-s/2)+\mathrm{sch}(x+s/2)]$, $A=\sqrt{|\beta/\gamma|}$ in stationary medium for $\beta = 0.05$, $\gamma = -1$; $\alpha, f' = 0$; $s = 10, 5$.

When stability condition [5] is not satisfied we observe in all cases a dispersion of pulses at evolution. Accounting for the effect of dissipation ($\alpha > 0$ in the GNLS equation), as in the case of the GKP equation, leads, on a level with general damping of the amplitude of the wave field, to a change in the structure of solitons with increasing the steepness of the leading fronts and elongating of tails.

## III. Conclusion

In conclusion, we have studied numerically the dynamics of the 1D and 2D soliton structures which is described by the GKP and GNLS equations of the BK system, in order to confirm the previously obtained analytically stability conditions for these models. The results fully confirmed the analytical calculations [1]-[3], [5], [6] and can be useful in investigations in areas such as plasma physics, hydrodynamics, physics of upper atmosphere and nonlinear optics.

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