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ON AN ANALOG OF THE M.G. KREIN THEOREM FOR MEASURABLE OPERATORS

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Abstract

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $\mu_t(T)$, $t > 0$, be a rearrangement of a τ-measurable operator T. Let us consider a τ-measurable operator A, such that $\mu_t(A) > 0$ for all $t > 0$ and assume that $\mu_{2t}(A)/\mu_t(A) \to 1$ as $t \to \infty$. Let a τ-compact operator S be so that the operator $I + S$ is right invertible, where I is the unit of $\mathcal{M}$. Then, for a τ-measurable operator B, such that $A = B(I + S)$, we have $\mu_t(A)/\mu_t(B) \to 1$ as $t \to \infty$. It is an analog of the M.G. Krein theorem (for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, theorem 11.4, ch. V [Gohberg I.C., Krein M.G. Introduction to the theory of linear nonselfadjoint operators. In: *Translations of Mathematical Monographs.* Vol. 18. Providence, R.I., Amer. Math. Soc., 1969. 378 p.] for τ-measurable operators.

Keywords: Hilbert space, von Neumann algebra, normal trace, τ-measurable operator, distribution function, rearrangement, τ-compact operator

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Introduction

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. In theorem 3.5, we prove an analog of the M.G. Krein theorem (for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, theorem 11.4, ch. V, [1]) for τ-measurable operators. We also describe asymptotics of the generalized singular numbers for a product of almost commuting τ-measurable operators.

1. Notation, definitions, and preliminaries

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{M}$. Let I be the unit of $\mathcal{M}$. Let $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. Let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called as follows: *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *finite* if $\varphi(X) < +\infty$ for all $X \in \mathcal{M}^+$; *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$; *normal* if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated with a von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A self-adjoint operator is affiliated with $\mathcal{M}$ if and only if all the projections from its spectral decomposition of unity belong to $\mathcal{M}$.

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X of everywhere dense in $\mathcal{H}$ domain $\mathcal{D}(X)$ and affiliated with $\mathcal{M}$ is said to be τ-*measurable* if there exists such a projection $P \in \mathcal{M}^{\text{pr}}$ for any $\varepsilon > 0$ that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a *-algebra under transition to the adjoint operator, multiplication by a scalar, and strong addition and multiplication operations defined as closure of the usual operations [2, 3].

If X is a closed densely defined linear operator affiliated with $\mathcal{M}$ and $|X| = \sqrt{X^*X}$, then the spectral decomposition $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$ and X belongs to $\widetilde{\mathcal{M}}$ if and only if there exists a number $\lambda \in \mathbb{R}$, such that $\tau(P^{|X|}((\lambda, +\infty))) < +\infty$. Let $\mu_t(X)$ denote the *rearrangement* of the operator $X \in \widetilde{\mathcal{M}}$, i.e., the nonincreasing right continuous function $\mu(X)\colon (0,\infty) \to [0,\infty)$ given by the formula

$$\mu_t(X) = \inf\{\|XP\| : \ P \in \mathcal{M}^{\text{pr}}, \quad \tau(P^{\perp}) \leq t\}, \quad t > 0.$$

Then, $\mu_t(X) = \inf\{s \geq 0 : \ \lambda_s(X) \leq t\}$, where $\lambda_s(X) = \tau(P^{|X|}((s,\infty)))$ is the distribution function of X. The set of τ-compact operators $\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \lim\limits_{t\to+\infty} \mu_t(X) = 0\}$ is an ideal in $\widetilde{\mathcal{M}}$ [4].

Lemma 1 (see [4–6]). *Let* $X, Y \in \widetilde{\mathcal{M}}$. *Then*

1) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$*;*
2) $\mu_{s+t}(X+Y) \leq \mu_s(X) + \mu_t(Y)$ *for all* $s, t > 0$*;*
3) $\mu_{s+t}(XY) \leq \mu_s(X)\mu_t(Y)$ *for all* $s, t > 0$*;*
4) $\mu_t(|X|^p) = \mu_t(X)^p$ *for all* $p, t > 0$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, i.e., the *-algebra of all linear bounded operators on $\mathcal{H}$, and $\tau = \text{tr}$ is the canonical trace, then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$. In this case, $\widetilde{\mathcal{M}}_0$ is the compact operators ideal on $\mathcal{H}$ and

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{+\infty}$ is a sequence of an operator X s-numbers [1]; here, χ_A is the indicator function of a set $A \subset \mathbb{R}$.

2. A generalization of the M.G. Krein theorem for τ-measurable operators

Lemma 2. *The following conditions are equivalent for a nonincreasing function* $f\colon (0,\infty) \to (0,\infty)$*:*

(i) *there exists* $\lim\limits_{t\to\infty} \dfrac{f(at)}{f(t)} = 1$ *for some number* $0 < a \neq 1$*;*

(ii) *there exists* $\lim\limits_{t\to\infty} \dfrac{f(bt)}{f(t)} = 1$ *for every number* $b > 0$.

Proof. (i)$\Rightarrow$(ii). We have

$$1 = \lim_{t\to\infty} \frac{f(at)}{f(t)} = \left[\lim_{t\to\infty} \frac{f(at)}{f(t)}\right]^{-1} = \lim_{t\to\infty} \left[\frac{f(at)}{f(t)}\right]^{-1} = \\ = \lim_{t\to\infty} \frac{f(t)}{f(at)} = \lim_{u\to\infty} \frac{f(a^{-1}u)}{f(u)}, \quad (1)$$

where $u = at$ for all $t > 0$. Hence, we assume that $a, b > 1$.

Case 1: $1 < b < a$. Then, we have

$$\frac{f(a^{-1}t)}{f(t)} \geq \frac{f(bt)}{f(t)} \geq \frac{f(at)}{f(t)} \quad \text{for all } \ t > 0$$

and lemma follows from (1) and the squeeze theorem.

Case 2: $1 < a < b$. Then, for $k \equiv \min\left\{n \in \mathbb{N} : \dfrac{b}{a^{n+1}} < a\right\}$ and for all $t > 0$, we have

$$\frac{f(a^{-1}t)}{f(t)} \geq \frac{f(bt)}{f(t)} = \frac{f(bt)}{f\left(\frac{b}{a}t\right)} \frac{f\left(\frac{b}{a}t\right)}{f\left(\frac{b}{a^2}t\right)} \cdots \frac{f\left(\frac{b}{a^k}t\right)}{f\left(\frac{b}{a^{k+1}}t\right)} \frac{f\left(\frac{b}{a^{k+1}}t\right)}{f(t)} \geq$$

$$\geq \frac{f(bt)}{f\left(\frac{b}{a}t\right)} \frac{f\left(\frac{b}{a}t\right)}{f\left(\frac{b}{a^2}t\right)} \cdots \frac{f\left(\frac{b}{a^k}t\right)}{f\left(\frac{b}{a^{k+1}}t\right)} \frac{f(at)}{f(t)}$$

and lemma follows from relations (1) and

$$\lim_{t\to\infty} \frac{f(bt)}{f\left(\frac{b}{a}t\right)} = \lim_{t\to\infty} \frac{f\left(\frac{b}{a}t\right)}{f\left(\frac{b}{a^2}t\right)} = \cdots = \lim_{t\to\infty} \frac{f\left(\frac{b}{a^k}t\right)}{f\left(\frac{b}{a^{k+1}}t\right)} = 1,$$

combined with theorem on the limit of product of functions and the squeeze theorem. Lemma is proved.

□

Example 1. 1) The conditions of lemma 2 hold if there exists $\lim_{t\to\infty} f(t) = x > 0$.

2) Let us consider $f(t) = \dfrac{1}{\log(1+t)}$ for all $t > 0$. Then, there exists $\lim_{t\to\infty} f(t) = x = 0$ and the conditions of lemma 2 also hold by the L'Hospital theorem for $\dfrac{f(2t)}{f(t)} = \dfrac{\log(1+t)}{\log(1+2t)} = \left\{\dfrac{\infty}{\infty}\right\}$ as $t \to \infty$. Induction helps us to prove the same result for n-iterated function $f_n(t) = \dfrac{1}{\log\log\cdots\log(e^{n-1}+t)}$ for all $n \in \mathbb{N}$ and $t > 0$.

3) If functions f, g satisfy the conditions of lemma 2, then, for the functions $f_{p*}(t) = f(pt)$, $f_{p+}(t) = f(t+p)$, $\psi_{f,p}(t) = \displaystyle\int_t^{t+p} f(u)du$, $f(t^p)$, f^p $(0 < p < \infty)$, $\log(1+f)$, $f+g$, $\dfrac{f}{g}$ (if $\dfrac{f}{g}$ is nonincreasing), and fg, the conditions of lemma 2 also hold.

We prove it for f_{p+}, $\psi_{f,p}$, $\log(1+f)$ and $f+g$. The case of $x = \lim_{t\to\infty} f(t) > 0$ is trivial. Let us put $x = 0$. Since

$$\frac{f(t+p)}{f(2t+2p)} \leq \frac{f(t+p)}{f(2t+p)} = \frac{f_p(t)}{f_p(2t)} \leq \frac{f(t+p/2)}{f(2t+p)} \quad \text{for all} \quad t > 0,$$

we can apply the squeeze theorem.

Since $pf(t+p) \leq \psi_{f,p}(t) \leq pf(t)$, we have for all $t > p$ the estimates

$$\frac{f(3t)}{f(t)} \leq \frac{f(2t+p)}{f(t)} = \frac{pf(2t+p)}{p\ f(t)} \leq \frac{\psi_{f,p}(2t)}{\psi_{f,p}(t)} \leq \frac{pf(2t)}{pf(t+p)} = \frac{f(2t)}{f(t+p)} \leq 1$$

and are able to apply the squeeze theorem.

We have $\log(1+u) = u + o(u)$ as $u \to 0$ and $f(2t) = f(t) + o(f(t))$ as $t \to \infty$. Therefore

$$\frac{\log(1+f(2t))}{\log(1+f(t))} = \frac{f(2t)+o(f(2t))}{f(t)+o(f(t))} = \frac{f(2t)+o(f(t))}{f(t)+o(f(t))} = 1 + o(f(t))$$

as $t \to \infty$. For $h = f + g$ we have $o(f(t)) + o(g(t)) = o(h(t))$ and

$$\frac{h(2t)}{h(t)} - 1 = \frac{f(2t) - f(t) + g(2t) - g(t)}{f(t)+g(t)} = \\ = \frac{o(f(t)) + o(g(t))}{f(t)+g(t)} = \frac{o(h(t))}{h(t)} = o(1) \quad \text{as} \quad t \to \infty.$$

4) Let us consider f, as in lemma 2, numbers $\alpha, \beta > 0$ and a nonincreasing function $g\colon (0,\infty) \to (0,\infty)$, so that $f(\alpha t) \le g(t) \le f(\beta t)$ for all $t > 0$. Then, for the function g, the conditions of lemma 1 also hold.

Lemma 3. *Let $\mathcal{J}$ be a left ideal in a unital algebra $\mathcal{A}$ and $S \in \mathcal{J}$ be so that the element $I + S$ is right invertible (i.e., there exists $T \in \mathcal{A}$ with $(I+S)T = I$). Then, $T = I + X$ for some $X \in \mathcal{J}$.*

Proof. Since $(I+S)T = I$, we have $T = I - ST \equiv I + X$ with $X \equiv -ST \in \mathcal{J}$. Lemma is proved.

□

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$ and $\tau(I) = +\infty$.

Proposition 1 (cf. lemma 3). *Let an isometry operator $U \in \mathcal{M}$ and a selfadjoint operator $A \in \widetilde{\mathcal{M}}$ be so that $I + A$ is invertible in $\widetilde{\mathcal{M}}$. Then, the following conditions are equivalent:*

(i) $U - A \in \widetilde{\mathcal{M}}_0$;

(ii) $I - A,\ I - U \in \widetilde{\mathcal{M}}_0$.

Proof. (i)⇒(ii). We have $U^* - A = (U - A)^* \in \widetilde{\mathcal{M}}_0$ and

$$-U^*A + AU = U^*(U - A) - (U^* - A)U \in \widetilde{\mathcal{M}}_0.$$

Therefore, $I - A^2 = (U^* - A)(U + A) - U^*A + AU \in \widetilde{\mathcal{M}}_0$ and $I - A = (I - A^2)(I + A)^{-1} \in \widetilde{\mathcal{M}}_0$. Thus, $I - U = I - A - (U - A) \in \widetilde{\mathcal{M}}_0$.

(ii)⇒(i). We have $U - A = (I - A) - (I - U) \in \widetilde{\mathcal{M}}_0$. The proposition is proved.

□

Theorem 1. *Let an operator $A \in \widetilde{\mathcal{M}}$ be such that $\mu_t(A) > 0$ for all $t > 0$ and assume that there exists $\lim\limits_{t\to\infty} \frac{\mu_{2t}(A)}{\mu_t(A)} = 1$. Let an operator $S \in \widetilde{\mathcal{M}}_0$ be so that the operator $I + S$ is right invertible in $\widetilde{\mathcal{M}}$. Then, for an operator $B \in \widetilde{\mathcal{M}}$, such that $A = B(I + S)$, there exists $\lim\limits_{t\to\infty} \frac{\mu_t(A)}{\mu_t(B)} = 1$.*

Proof. Let a number $\varepsilon > 0$ be arbitrary and let a number $t_1 > 0$ be such that $\mu_{t/3}(S) < \varepsilon$ for $t \geq t_1$. Then, by items 2) and 3) of lemma 1, we have the following estimates for all $t \geq t_1$:

$$\begin{aligned} \mu_t(A) = \mu_t(B + BS) \leq \mu_{t/3}(B) + \mu_{2t/3}(BS) \leq \\ \leq \mu_{t/3}(B) + \mu_{t/3}(B)\mu_{t/3}(S) < \\ < (1+\varepsilon)\mu_{t/3}(B). \end{aligned} \tag{2}$$

Let an operator $T \in \widetilde{\mathcal{M}}$ be such that $(I + S)T = I$. Then, $T = I + X$ with some $X \in \widetilde{\mathcal{M}}_0$, see lemma 3. Since

$$AT = B(I + S)T = B = A(I + X),$$

for number $t_2 > 0$ with $\mu_{t/3}(X) < \varepsilon$ for $t \geq t_2$, we obtain, analogously to estimates (2), the relation

$$\mu_t(B) < (1+\varepsilon)\mu_{t/3}(A) \quad \text{for all} \quad t \geq t_2. \tag{3}$$

Let a number $t_3 > 0$ be such that

$$1 \leq \frac{\mu_{t/9}(A)}{\mu_{t/3}(A)} < 1 + \varepsilon \quad \text{for all} \quad t \geq t_3,$$

see lemma 2. Let us put $t_0 = \max\{t_1, t_2, t_3\}$. From (2) and (3) we obtain for all $t > t_0$

$$\mu_t(A) < (1+\varepsilon)\mu_{t/3}(B) < (1+\varepsilon)^2\mu_{t/9}(A),$$

hence,

$$1 \leq \frac{\mu_t(A)}{\mu_{t/3}(A)} < (1+\varepsilon)\frac{\mu_{t/3}(B)}{\mu_{t/3}(A)} < (1+\varepsilon)^2\frac{\mu_{t/9}(A)}{\mu_{t/3}(A)} < (1+\varepsilon)^3.$$

Therefore,

$$1 < (1+\varepsilon)\frac{\mu_{t/3}(B)}{\mu_{t/3}(A)} < (1+\varepsilon)^3 \quad \text{for all} \quad t > t_0.$$

The theorem is proved. □

Corollary 1. *Let an operator* $A \in \widetilde{\mathcal{M}}$ *be such that* $\mu_t(A) > 0$ *for all* $t > 0$ *and assume that there exists* $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(A)}{\mu_t(A)} = 1$. *Let an operator* $S \in \widetilde{\mathcal{M}}_0$ *be so that the operator* $I + S$ *is left invertible in* $\widetilde{\mathcal{M}}$. *Then, for an operator* $B \in \widetilde{\mathcal{M}}$, *such that* $A = (I + S)B$, *there exists* $\lim\limits_{t\to\infty} \dfrac{\mu_t(A)}{\mu_t(B)} = 1$.

Proof. We have $S^* \in \widetilde{\mathcal{M}}_0$ and since $(XY)^* = Y^*X^*$ for all $X, Y \in \widetilde{\mathcal{M}}$, the operator $I + S^*$ is right invertible in $\widetilde{\mathcal{M}}$. Therefore, $A^* = B^*(I + S^*)$. Then, we apply theorem 1 for the operators A^*, B^*, S^* and recall item 1) of lemma 1. The corollary is proved.

□

Example 2. Let operators $X, Y \in \widetilde{\mathcal{M}}$ be almost commuting, i.e., the commutator $[X, Y] = XY - YX \in \widetilde{\mathcal{M}}_0$. Let us put $K = [X, Y]$ and let the operator YX possess a right inverse $T \in \widetilde{\mathcal{M}}$. Hence, $XY = YX(I + TK)$. Since the operator YX is right invertible by item 3) of lemma 1, we have $1 = \mu_t(I) = \mu_t(YXT) \leq \mu_{t/2}(YX)\mu_{t/2}(T)$ for all $t > 0$. Hence, $\mu_t(YX) > 0$ for all $t > 0$. Now, if the operator $I + TK$ possess a right inverse $R \in \widetilde{\mathcal{M}}$ (then $XYR = YX(I+TK)R = YX$ and by item 3) of lemma 1, we have $0 < \mu_t(YX) \le \mu_{t/2}(XY)\mu_{t/2}(R)$ for all $t > 0$; hence, $\mu_t(XY) > 0$ for all $t > 0$) and there exists $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(XY)}{\mu_t(XY)} = 1$, then there exists $\lim\limits_{t\to\infty} \dfrac{\mu_t(XY)}{\mu_t(YX)} = 1$ by theorem 1. For any normal operators $X, Y \in \widetilde{\mathcal{M}}$, we have $\mu_t(XY) = \mu_t(YX)$ for all $t > 0$ [7, corollary 3.6].

Remark 1. In theorem 1 and corollary 1 by item 4) of lemma 1, there exists $\lim\limits_{t\to\infty} \dfrac{\mu_t(|A|^p)}{\mu_t(|B|^p)} = 1$ for every $p > 0$. For $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, the condition "there exists $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(A)}{\mu_t(A)} = 1$" also appeared in [8].

Example 3. Let (Ω, ν) be a measure space and $\mathcal{M}$ be the von Neumann algebra of multiplicator operators M_f by functions f from $L_\infty(\Omega, \nu)$ on a space $L_2(\Omega, \nu)$. The algebra $\mathcal{M}$ containes no compact operators $\Leftrightarrow$ the measure ν has no atoms [9, theorem 8.4]. Let $\mathcal{M} = L_\infty(0, \infty)$ and $\mathcal{H} = L_2(0, \infty)$. Then, for any right continuous nonincreasing function $f\colon (0, \infty) \to (0, \infty)$, we have $\mu_t(M_f) = f(t)$ for all $t > 0$, see definition 2.2, ch. II, [10]. Example 1 shows that the set of multiplicator operators M_f, such that there exists $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(M_f)}{\mu_t(M_f)} = 1$, is relatively rich.

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Об аналоге теоремы М.Г. Крейна для измеримых операторов

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Аннотация

Пусть алгебра фон Неймана операторов $\mathcal{M}$ действует в гильбертовом пространстве $\mathcal{H}$ и τ – точный нормальный полуконечный след на $\mathcal{M}$. Пусть $\mu_t(T)$, $t > 0$, – перестановка τ-измеримого оператора T. Пусть τ-измеримый оператор A такой, что $\mu_t(A) > 0$ для всех $t > 0$ и пусть $\mu_{2t}(A)/\mu_t(A) \to 1$ при $t \to \infty$. Пусть τ-компактный оператор S такой, что оператор $I + S$ является обратимым справа, где I – единица алгебры $\mathcal{M}$. Тогда для τ-измеримого оператора B такого, что $A = B(I + S)$, имеем $\mu_t(A)/\mu_t(B) \to 1$ при $t \to \infty$. Это является аналогом теоремы М.Г. Крейна (для $\mathcal{M} = \mathcal{B}(\mathcal{H})$ и $\tau = \mathrm{tr}$ (теорема 11.4, гл. V, [Гохберг И.Ц., Крейн М.Г. Введение в теорию линейных несамосопряженных операторов. – М.: Наука, 1965. – 448 с.]), для τ-измеримых операторов.

Ключевые слова: гильбертово пространство, алгебра фон Неймана, нормальный след, τ-измеримый оператор, функция распределения, перестановка, τ-компактный оператор

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