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A Note about Torsional Rigidity and Euclidean Moment of Inertia of Plane Domains

R. G. Salakhudinov*

(Submitted by F. G. Avkhadiev)

N. I. Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga region) Federal University, ul. Kremlevskaya 18, Kazan, Tatarstan, 420008 Russia

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Abstract—Denote by $\mathbf{P}(G)$ the torsional rigidity of a simply connected plane domain G, and by $\mathbf{I}_2(G)$ the Euclidean moment of inertia of G. In 1995 F.G. Avkhadiev proved that $\mathbf{P}(G)$ and $\mathbf{I}_2(G)$ are comparable quantities in sense of Pólya and Szegö. Moreover, it was shown that the ratio $\mathbf{P}(G)\,/\mathbf{I}_2(G)$ belongs to the segment $[1, 64]$. We investigate the following conjecture $\mathbf{P}(G) \geq 3\mathbf{I}_2(G)$, where G is a simply connected domain. We prove that the conjecture is true for polygonal domains circumscribed about a circle. For convex domains we show sharp isoperimetric inequalities, which justify the conjecture, in particular, we prove that $\mathbf{P}(G) > 2\mathbf{I}_2(G)$. Some aspects of approximate formulas for $\mathbf{P}(G)$ are also discussed.

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1. INTRODUCTION AND SOME CONJECTURES

In analytical studies of steady field problems of the theory of elasticity [1] and microfluid mechanics [2] a boundary value problem

$$\triangle \mathrm{u} = -2 \ \text{ in } \ G, \quad \mathrm{u} = 0 \ \text{ on } \ \partial G \tag{1}$$

is solved for complex-shaped domains G, here ∂G is the boundary curve of G. A classical result of the theory in partial differential equations is that there is a unique solution of the boundary value problem (1) for a simply connected domain G. Isoperimetric estimates in these solutions (see, for instance, [3, 4]) require evaluation of the functional

$$\mathbf{P}(G)^{1)} := 2\int\limits_G \mathrm{u}(x, G)\mathrm{dA}. \tag{2}$$

This functional is called the *torsional rigidity* of G in the theory of elasticity, and the *flow rate* in hydrodynamics of pipe flows.

Investigations of the physically important boundary value problem (1), and the physical functional (2) go back to works of B. de Saint-Venant [5], and Lord Rayleigh (see [3, 6]). Estimates of $\mathbf{P}(G)$, using different geometrical or (and) physical characteristics of G, intensively studied by mathematicians since the middle of XX century, when their interest was stimulated by the famous monograph of G. Pólya and G. Szegö [3]. Then a number of works [6] is growing rapidly in this area. There are a number of isoperimetric inequalities [6] for $\mathbf{P}(G)$ in this branch of mathematical physics.

*E-mail: rsalakhud@gmail.com

1)Throughout the paper we will use the bold face for notations of functionals.

We point out inequalities which, using a unique functional of a domain, give estimates of the torsional rigidity from the upper and the down sides. Note that most of inequalities which includes the torsional rigidity are one sided estimates, i. e. we cannot show an opposite inequality in the same class of domains. For instance, a famous and important inequality in the theory of elasticity is the Saint-Venant—Pólya inequality [4]

$$\mathbf{P}(G) \leq (2\pi)^{-1}\mathbf{A}(G)^2, \tag{3}$$

where $\mathbf{A}(G)$ is the area of G. The latter inequality is valid in the class of simply connected domains. Note that a disk is a unique extremal domain in inequality (3), i. e. the Saint-Venant—Pólya inequality turns into the equality if and only if G is a disk. It is shown in [3] that in the same class of domains an opposite inequality $\mathrm{C}\mathbf{A}(G)^2 \leq \mathbf{P}(G)$ is possible if and only if $\mathrm{C} \equiv 0$. In particular, we cannot construct an effective approximation formula for the torsional rigidity in terms of $\mathbf{A}(G)^2$.

Since the middle of XX century an important step in the investigation of the torsional rigidity was made. It was found that for solving some problems of mathematical physics, the classical geometric characteristics of a domain (the area, the length of the boundary, and many other) are insufficient. Let

$$\mathbf{I}_p(G) := \int_G \rho(x,G)^p \mathrm{dA} \tag{4}$$

be the p-order Euclidean moment of G with respect to its boundary, where $\rho(x,G)$ is the distance function from a point $x (x \in G)$ to the boundary ∂G. In 1995 Avkhadiev [7] proved that if G is a simply connected domain, then the torsional rigidity and the Euclidean moment of inertia $\mathbf{I}_2(G)$ are comparable quantities (in the sense of Pólya and Szegö [3, p. 112]). Moreover, the following chain of inequalities

$$\mathbf{I}_2(G) \leq \mathbf{P}(G) \leq 64\mathbf{I}_2(G) \tag{5}$$

holds. The both constants '1' and '64' in (5) are not sharp. As far us we know, there is not a suitable hypothesis for the best constant and for extremal domains in the right inequality of (5). It is noteworthy that the extremal domains are not convex.

On the other hand, the lower bound of $\mathbf{P}(G)$ in (5) is a consequence of the pointwise estimate $2\mathrm{u}(x,G) \geq \rho(x,G)^2 \ (x \in G)$. In 2001 Salakhudinov [8] improved the left inequality in (5) to the inequality

$$(3/2)\mathbf{I}_2(G) \leq \mathbf{P}(G), \tag{6}$$

which is still not sharp. However, it was noted in [8] that (6) is a direct corollary of the exact isoperimetric inequality

$$(3/2)\int_G \mathrm{R}(x,G)^2 \mathrm{dA} \leq \mathbf{P}(G),$$

where $\mathrm{R}(x,G)$ is the conformal radius at a point x of G. The latter inequality turns into equality if and only if G is a disk (see also [9]).

Now let us recall well-known inequalities [10] $\rho(x,G) \leq \mathrm{R}(x,G) \leq 4\rho(x,G) \ (x \in G)$ for simply connected domains in the geometric function theory. From the latter two relations we formulate the following almost obvious conjecture.

Conjecture 1. *Let G be a simply connected domain with a finite torsional rigidity, then the following inequality*

$$3\mathbf{I}_2(G) \leq \mathbf{P}(G) \tag{7}$$

holds. The equality case holds if and only if G is a disk.

We cannot prove Conjecture 1 in this paper, but below we obtain results, which have to be considered as a certain justification of the conjecture.

But above all, let us restrict your consideration to the case of convex domains, then more inequalities are arisen. Indeed, the functional (4) was first considered by Makai [11] with a relation to the torsion problem. It was proved (see also [7]) that

$$\mathbf{P}(G) \leq 4\mathbf{I}_2(G), \tag{8}$$

where the constant '4' is sharp, but extremal domains are degenerate, such as a "needle".

Before the inequalities (5) the same problem of two sided estimates for the torsional rigidity in the class of convex domains was closed in 1965 by Makai [12]. As a corollary of (8) he added the upper bound of the torsional rigidity

$$\mathbf{P}(G) \leq (4/3)\mathbf{A}(G)\boldsymbol{\rho}(G)^2 \tag{9}$$

to the Pólya–Szegö inequality [3]:

$$(1/2)\mathbf{A}(G)\boldsymbol{\rho}(G)^2 \leq \mathbf{P}(G). \tag{10}$$

Here $\boldsymbol{\rho}(G)$ is the inradius of G, that is $\boldsymbol{\rho}(G) := \sup_{x\in G}\rho(x,G)$. The both constants '$1/2$' and '$4/3$' are sharp. Extremal domains in (9) are the same as in (8), and (10) turns in to the equality, for instance, for a disk.

Now, let's make a few comments. Firstly, Conjecture 1 has close relations with Pólya–Szegö's inequality (10). Indeed, in [13] it was proved the isoperimetric inequality

$$\mathbf{I}_2(G) \geq (1/6)\left(\mathbf{A}(G)\boldsymbol{\rho}(G)^2 + \mathbf{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right), \tag{11}$$

where G is a convex domain, and $\mathbf{l}(\boldsymbol{\rho}(G))$ is the length of the level curve of $\rho(x,G)$ located at the distance $\boldsymbol{\rho}(G)$ from the boundary ∂G. The case of equality in the latter inequality holds for polygonal domains circumscribed about a circle, as well for a disk, and for some other domains described in [13]. It is clear that (7) and (11) imply (10). Indeed, it follows a little bit stronger inequality

$$(1/2)\left(\mathbf{A}(G)\boldsymbol{\rho}(G)^2 + \mathbf{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right) \leq \mathbf{P}(G),$$

which remains unproven if $\mathbf{l}(\boldsymbol{\rho}(G)) \neq 0$.

Secondly, in [14] another isoperimetric inequality

$$\mathbf{A}(G)^2 - 6\pi\mathbf{I}_2(G) \geq \left(\mathbf{A}(G) - \pi\boldsymbol{\rho}(G)^2\right)^2 \tag{12}$$

was proved for simply connected domains with bounded area. The equality in (12) holds if and only if G is a Bonnesen type domain. We recall that Bonnesen type domains are extremal domains in the Bonnesen isoperimetric inequality [15]. Combining the latter inequality with (10), we obtain

$$\mathbf{P}(G) \geq (3/2)\mathbf{I}_2(G) + (\pi/4)\boldsymbol{\rho}(G)^4, \tag{13}$$

where G is a convex domain, and the equality case holding for a disk. As a corollary of (5) we cannot generalize Pólya–Szegö's inequality for simply connected domains (for more details see [7]), but we can state the following conjecture:

Conjecture 2. *Let G be a simply connected domain with a finite torsional rigidity, then the following inequality $(3/2)\mathbf{I}_2(G) + (\pi/4)\boldsymbol{\rho}(G)^4 \leq \mathbf{P}(G)$ holds. The inequality turns into equality if and only if G is a disk.*

It is clear that Conjecture 1 is an isoperimetric rectification of inequality (6). On the other hand, from the monotonicity of the functional $\mathbf{I}_2(G)$ with respect to enlargement of the domain, we have

$$\mathbf{I}_2(G) \geq (\pi/6)\boldsymbol{\rho}(G)^4. \tag{14}$$

Therefore, Conjecture 2 follows from Conjecture 1. However, we know now that the Conjecture 2 is true for convex domains. Finally, some remarks follow from our main results stated below.

It is worth to note that the same problems are considered in higher dimensions (see, for instance, [16, 17]), but we concentrate on the two dimensional case because of its practical applications.

2. MAIN RESULTS AND THEIR PROOFS

Let us start from some definitions. Let G be a convex domain, and let $\mathbf{l}(\boldsymbol{\rho}(G)) \neq 0$. Then a rectangle with sides $2\boldsymbol{\rho}(G)$ and $\mathbf{l}(\boldsymbol{\rho}(G))$ is a part of G (see [13]). Removing the rectangle from the domain G, we can glue together the two remaining parts of G. So, we get another convex domain G_0. It is clear that $\boldsymbol{\rho}(G_0) = \boldsymbol{\rho}(G)$, and $\mathbf{l}(\boldsymbol{\rho}(G_0)) = 0$. We call the domain G_0 a *compression* of G. On the other side, the domain G is a *stretch* of G_0 so that $\boldsymbol{\rho}(G) = \boldsymbol{\rho}(G_0)$. It is worth noting that we cannot stretch any convex domain, for instance, it is impossible to stretch a triangle or a regular pentagon.

Theorem 1. *Let G be a polygonal domain circumscribed around a circle or be a stretch of a polygonal domain circumscribed around a circle. Then*

$$\mathbf{P}(G) \geq (1/4)(p+1)(p+2)\mathbf{I}_p(G)\boldsymbol{\rho}(G)^{2-p}, \tag{15}$$

where $-1 \leq p < +\infty$. The equality holds if G is a disk. Moreover, if $\mathbf{l}(\boldsymbol{\rho}(G)) \neq 0$, then (15) *turns into a strict inequality.*

For $p = 0$ the inequality (16) turns into Pólya–Szegö's inequality. If we put $p = 2$, then we obtain (7), that is Conjecture 1 valid for polygonal domains circumscribed around a circle. The case $p = -1$ corresponds to the following inequality $\mathbf{P}(G) \geq \mathbf{L}(G)\boldsymbol{\rho}(G)^3/4$, where $\mathbf{L}(G)$ is the length of the boundary of G.

Theorem 2. *Let G be a bounded convex domain in the plane. Then*

$$\mathbf{P}(G) \geq 2\mathbf{I}_2(G) + (\pi/6)\boldsymbol{\rho}(G)^4. \tag{16}$$

The equality holds if G is a disk. Furthermore, for domains with $\mathbf{l}(\boldsymbol{\rho}(G)) \neq 0$ (16) *turns into a strict inequality.*

It follows from (14) that the latter inequality is stronger than (13), and still it is weaker than (7) for convex domains. Under the same argumentation as above, we can state the next conjecture.

Conjecture 3. *Let G be a simply connected domain with bounded the torsional rigidity, then the following inequality $2\mathbf{I}_2(G) + (\pi/6)\boldsymbol{\rho}(G)^4 \leq \mathbf{P}(G)$ holds. The equality case holds if and only if G is a disk.*

Let us continue to collect remarks begun in the previous section. We prove Conjecture 1 for a subset of convex domains. Also, we propose a conjecture, which is weaker than Conjecture 1, but stronger than Conjecture 2. Again, our conjecture holds for convex domains.

Proofs of Theorem 1 and Theorem 2. We start from standard definitions in the theory of estimates in subsets bounded by level curves. Denote by

$$G(\mu) := \{x \in G \,|\, \rho(x, G) > \mu\}, \quad \mathbf{a}(\mu) \equiv \mathbf{A}(G(\mu)) := \int\limits_{G(\mu)} \mathrm{dA}, \tag{17}$$

and $\mathbf{l}(\mu) := \mathbf{L}(G(\mu))$, where $\mathbf{L}(G(\mu))$ is the length of the boundary curve of $G(\mu)$. We first recall some results from the torsion theory. We will use the distance function to the boundary of a domain as a reference function. It was proved by Pólya–Szegö [3, p. 102] that

$$\mathbf{P}(G) \geq 4 \int\limits_0^{\boldsymbol{\rho}(G)} \frac{\mathbf{a}(\mu)^2}{\mathbf{l}(\mu)} \mathrm{d}\mu. \tag{18}$$

Denote by $\mathbf{P}^-(G)$ the functional in the right hand side of the latter inequality.

Let $\mathcal{C}(G)$ be the space of all real-valued functions $f(x)$, vanishing on ∂G, continuous in G, and piecewise continuously differentiable in G, for which the Dirichlet integral is finite. Kohler-Jobin [18] noted that $\mathbf{P}^-(G)$ is the exact value of the maximum principle

$$\max_{f(x)\in\mathcal{C}(G),\, f(x)=\varphi(\rho(x,G))} \left\{ 4\int\limits_G f \mathrm{dA} - \int\limits_G |\nabla f|^2 \mathrm{dA} \right\}, \tag{19}$$

where $\varphi(t)$ denotes a real-valued function of the real value t. Furthermore, the function

$$v(\bar\rho) := 2\int_0^{\bar\rho} \frac{\mathbf{a}(\mu)}{\mathbf{l}(\mu)}\mathrm{d}\mu \tag{20}$$

solves the maximal principle (19), here $\bar\rho = \rho(x, G)$. In addition,

$$\mathbf{P}^-(G) = 2\int_G v(\rho(x,G))\mathrm{dA} = \int_G |\nabla v(\rho(x,G))|^2 \mathrm{dA}. \tag{21}$$

As Kohler-Jobin notes, if we replace the reference function $\rho(x, G)$ by the warping function $\mathrm{u}(x, G)$, then $\mathbf{P}^-(G)$ turns into the torsional rigidity $\mathbf{P}(G)$. In particular,

$$\mathbf{P}(G) \geq \mathbf{P}^-(G). \tag{22}$$

Therefore, follow Kohler-Jobin we call $\mathbf{P}^-(G)$ the modified torsional rigidity of G with respect to the function $\rho(x, G)$.

To prove the theorems we apply the next assertion, which was proved in [13].

Lemma 1. *Let G be a convex domain with a finite area. Then the following inequality*

$$\mathbf{A}(G) \geq (1/2)[\mathbf{L}(G) + \mathbf{l}(\boldsymbol{\rho}(G))]\boldsymbol{\rho}(G)$$

holds. The inequality turns into equality, for instance, if the boundary of G is a polygon circumscribed about a circle.

From the assertion follows the weaker inequality

$$\mathbf{A}(G) \geq \mathbf{L}(G)\boldsymbol{\rho}(G)/2, \tag{23}$$

which we will apply below. It is clear that $G(\mu)$ are convex domains for $0 \leq \mu < \boldsymbol{\rho}(G)$, and $\boldsymbol{\rho}(G(\mu)) = \boldsymbol{\rho}(G) - \mu$. Now, applying (23) for domains $G(\mu)$, and inserting it into (20), we obtain

$$v(\bar\rho) \geq \int_0^{\bar\rho} \boldsymbol{\rho}(G(\mu))\mathrm{d}\mu = \boldsymbol{\rho}(G)\bar\rho - \bar\rho^2/2.$$

This leads to the weaker inequality $v(\bar\rho) \geq \bar\rho^2/2$. Applying the definition of Lebesque's integral to (21), we obtain

$$\mathbf{P}^-(G) = -2\int_0^{\boldsymbol{\rho}(G)} v(\bar\rho)\mathrm{d}\mathbf{a}(\bar\rho). \tag{24}$$

This representation together with the latter inequality leads to the inequality

$$\mathbf{P}^-(G) \geq -\int_0^{\boldsymbol{\rho}(G)} \bar\rho^2 \mathrm{d}\mathbf{a}(\bar\rho) = \mathbf{I}_2(G).$$

From (22), it follows that the obtained inequality stronger than the left hand side inequality in (5).

On the other side, taking into account that $\mathbf{l}(\boldsymbol{\rho}(G(\mu))) = \mathbf{l}(\boldsymbol{\rho}(G))$, and $\mathbf{l}(\mu)$ is a decreasing function, from Lemma 1 we get

$$v(\bar\rho) \geq \int_0^{\bar\rho} \left(\boldsymbol{\rho}(G(\mu)) + \frac{\mathbf{l}(\boldsymbol{\rho}(G(\mu)))}{\mathbf{l}(\mu)}\boldsymbol{\rho}(G(\mu))\right)\mathrm{d}\mu \geq \left(1 + \frac{\mathbf{l}(\boldsymbol{\rho}(G))}{\mathbf{L}(G)}\right)\int_0^{\bar\rho} (\boldsymbol{\rho}(G) - \mu)\mathrm{d}\mu$$
$$= \left(1 + \frac{\mathbf{l}(\boldsymbol{\rho}(G))}{\mathbf{L}(G)}\right)\left(\boldsymbol{\rho}(G)\bar\rho - \frac{\bar\rho^2}{2}\right).$$

Inserting the latter inequality into (24), we have

$$\mathbf{P}^-(G) \geq \left(1 + \frac{\mathbf{l}(\boldsymbol{\rho}(G))}{\mathbf{L}(G)}\right)\left(-2\boldsymbol{\rho}(G)\int\limits_0^{\boldsymbol{\rho}(G)} \bar{\rho}\mathrm{d}\mathbf{a}(\bar{\rho}) + \int\limits_0^{\boldsymbol{\rho}(G)} \bar{\rho}^2\mathrm{d}\mathbf{a}(\bar{\rho})\right)$$

$$= \left(1 + \frac{\mathbf{l}(\boldsymbol{\rho}(G))}{\mathbf{L}(G)}\right)(2\boldsymbol{\rho}(G)\mathbf{I}_1(G) - \mathbf{I}_2(G)), \tag{25}$$

here $\mathbf{I}_1(G)$ is the stationary Euclidean moment of G with respect to its boundary.

Now we apply the following inequality

$$\mathbf{I}_2(G) - \frac{\pi\boldsymbol{\rho}(G)^4}{6} \leq \frac{2\boldsymbol{\rho}(G)}{3}\left(\mathbf{I}_1(G) - \frac{\pi\boldsymbol{\rho}(G)^3}{3}\right), \tag{26}$$

which was proved in [14, see Theorem 2 for $q = 2$]. This expression, together with (25), yields the inequality

$$\mathbf{P}^-(G) \geq \left(1 + \frac{\mathbf{l}(\boldsymbol{\rho}(G))}{\mathbf{L}(G)}\right)\left(2\mathbf{I}_2(G) + \frac{\pi\boldsymbol{\rho}(G)^4}{6}\right).$$

From this inequality, and (22) follow the desired inequality (16). Note that the inequality (26) valid for simply connected domains with the bounded stationary Euclidean moment, and it turns into equality if and only if G is a Bonnesen-type domain. Therefore, a disk belongs to the intersection of extremal domains in the inequalities (23), and (26).

Now, let G is a polygonal domain circumscribed around a circle. Then the inequality (23) turns into equality, and we can compute $\mathbf{P}^-(G)$ in terms of Euclidean moments of G, that is

$$\mathbf{P}^-(G) = 2\boldsymbol{\rho}(G)\mathbf{I}_1(G) - \mathbf{I}_2(G). \tag{27}$$

Since ∂G is a polygon circumscribed around a circle, then the following formula holds

$$\mathbf{a}(\bar{\rho}) = \mathbf{A}(G) - \mathbf{L}(G)\bar{\rho} + \bar{\rho}^2\sum_{i=1}^{n}\cot(\alpha_i/2),$$

where $0 \leq \bar{\rho} \leq \boldsymbol{\rho}(G)$, and α_i $(i = \overline{1,n})$ denote the angles at vertices of the polygon G. This expression, together with the definition of Lebesque's integral, leads to the equality

$$\mathbf{I}_q(G) = \frac{(p+1)(p+2)}{(q+1)(q+2)}\mathbf{I}_p(G)\boldsymbol{\rho}(G)^{q-p},$$

where $p, q > -1$. Inserting the latter equality with $q = 1$, and $q = 2$ into (27), we obtain (15). This complete the proof of Theorem 1.

In the conclusion we note that the Euclidean moments of polygonal domains circumscribed around a circle are linearly connected up to the factor $\boldsymbol{\rho}(G)^{p-q}$, therefore the Euclidean moments can be considered as the unique geometric functional of G for all p $(p \geq -1)$. $\square$

3. CONCLUSION

To calculate $\mathbf{P}(G)$ in practice, we can apply special computer programs to solve the boundary value problem (1), or we can make an experiment to get experimental data, for instance, it is a usual way to evaluate the flow rate in microfluid mechanics. Any way, if we cannot get an exact solution of (1), then we obtain an approximate value of $\mathbf{P}(G)$. As a result of an approximate calculation, we usually have a number with four significant digits after the decimal point (see, for instance, Table 1). How to estimate this approximation? Below we collect answers on this question.

For a convex domain G from the inequalities (9), and (10) follow an approximation formula

$$\mathbf{P}(G) \approx (11/12)\mathbf{A}(G)\boldsymbol{\rho}(G)^2. \tag{28}$$

Thus, the ratio $\mathbf{P}(G)\big/\big((11/12)\mathbf{A}(G)\boldsymbol{\rho}(G)^2\big)$ belongs to the interval $[6/11, 16/11]$ with the length $10/11 \approx 0,9091$. It is clear that the length of the interval we can consider as an efficiency criterion of (28). Indeed, the length of the interval corresponds to the spread of values of $\mathbf{P}(G)\big/\big((11/12)\mathbf{A}(G)\boldsymbol{\rho}(G)^2\big)$. Therefore, an approximation formula for $\mathbf{P}(G)$ with a smaller spread of values gives a better approach to the torsional rigidity in a class of domains.

On the other hand, we can use (28) to test experimental data. Indeed, if P is an experimental data of $\mathbf{P}(G)$, and the value of $\mathrm{P}\big/\big((11/12)\mathbf{A}(G)\boldsymbol{\rho}(G)^2\big)$ is outside of the mentioned above interval, then we consider the value P as a rough approximation of $\mathbf{P}(G)$ for later use. The same is true for the approximate formulas given below. Thus, we have simple and effective methods of rejection.

As a corollary of Theorem 2, and inequality (8) we can state another approximation formula $\mathbf{P}(G) \approx 3\mathbf{I}_2(G)$ in the same class of domains. Note that, in this case the ratio $\mathbf{P}(G)/(3\mathbf{I}_2(G))$ lies in the interval $(2/3, 4/3]$ with the length $2/3 \approx 0.6667$ that is less than $10/11$. It is interesting to note that we can consider the latter formula as an approximation of $\mathbf{I}_2(G)$, for example, when G is an ellipse. Indeed, in this case the problem (1) has the exact solution, but there is not a precise formula for the distance function to the boundary.

In the same way, applying Theorem 1 together with (8), we obtain

$$\mathbf{P}(G) \approx 3.5\mathbf{I}_2(G), \tag{29}$$

where G is a polygonal domain circumscribed around a circle. The corresponding interval becomes $[6/7, 8/7]$ with the best possible length $2/7 \approx 0.2857$.

Let us show the corresponding results in the class of simply connected domains. We have $\mathbf{P}(G) \approx (131/4)\mathbf{I}_2(G)$, the corresponding ratio belongs to the interval $(6/131, 256/131]$ with the length $250/131 \approx 1.9084$.

On the other hand, it follows from Conjecture 1, and (8) that the approximation formula (29) may be valid in a wider class of domains, but this is not the case for the class of simply connected domains.

Approximation formulas are illustrated in Table 1 (see Appendix B), where we underlined the closest values to the value of $\mathbf{P}(G)$. It is noteworthy that a best formula to approximate $\mathbf{P}(G)$ in a class of domains and an approximate formula, which gives the closest value to $\mathbf{P}(G)$, are different concepts.

In this section we applied the formula of arithmetic mean in order to construct the approximations of $\mathbf{P}(G)$. In the same way we can use the formulas of geometric mean, or harmonic mean, as well as another approximation on a segment.

Results around Conjecture 1 are summarized in Table 2 (see Appendix B).

Additionally, the following chain of implications

$$\begin{gathered}\{3\mathbf{I}_2(G) \le \mathbf{P}(G)\} \Rightarrow \big\{2\mathbf{I}_2(G) + \pi\boldsymbol{\rho}(G)^4/4 \le \mathbf{P}(G)\big\}\\ \Rightarrow \big\{3\mathbf{I}_2(G)/2 + \pi\boldsymbol{\rho}(G)^4/4 \le \mathbf{P}(G)\big\} \Rightarrow \{3\mathbf{I}_2(G) < 2\mathbf{P}(G)\}\end{gathered}$$

holds in a fixed class of simply connected domains, where the implication {inequality 1} $\Rightarrow$ {inequality 2} means that the inequality 2 follows from the inequality 1. Also, the following chain

$$\begin{gathered}\{3\mathbf{I}_2(G) \le \mathbf{P}(G)\} \Rightarrow \big\{\mathbf{A}(G)\boldsymbol{\rho}(G)^2 \le 2\mathbf{P}(G)\big\}\\ \Rightarrow \big\{3\mathbf{I}_2(G)/2 + \pi\boldsymbol{\rho}(G)^4/4 \le \mathbf{P}(G)\big\} \Rightarrow \{3\mathbf{I}_2(G) < 2\mathbf{P}(G)\}\end{gathered}$$

holds in a fixed class of convex domains.

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Appendix A

$\mathrm{u}(x, G)$ = the warping function of G at a point x in the elasticity theory, or the velocity of a fluid at the point x in hydrodynamic interpretation;

$\mathbf{P}(G)$ = the torsional rigidity of G, or the flow rate of G;

$\mathbf{A}(G)$ = the area of G;

$\rho(x, G)$ = the distance function from a point x to the boundary ∂G;

$\boldsymbol{\rho}(G)$ = the inradius of G, that is, the radius of the largest disk contained in G;

$\mathbf{I}_1(G)$ = the stationary Euclidean moment of G;

$\mathbf{I}_2(G)$ = the Euclidean moment of inertia of G;

$\mathbf{I}_p(G)$ = the p-order Euclidean moment of G with respect to its boundary;

$\mathrm{R}(x, G)$ = the conformal radius at a point x of G;

$\mathbf{L}(G)$ = the length of the boundary of G;

$\mathbf{l}(\boldsymbol{\rho}(G))$ = the length of the level curve of $\rho(x, G)$ located at the distance $\boldsymbol{\rho}(G)$ from the boundary ∂G;

$G(\mu)$ = the level set of the distance function to the boundary at the distance μ;

$\mathbf{a}(\mu)$ = the area of $G(\mu)$;

$\mathbf{l}(\mu)$ = the length of the boundary curve of $G(\mu)$;

$v(\bar{\rho})$ = the auxiliary function, which solve a variational problem;

$\mathbf{P}^{-}(G)$ = the modified torsional rigidity of G with respect to the function $\rho(x, G)$.

Appendix B

Table 1

Domain	Approximation formulas			$\mathbf{P}(G)$
	$11\mathbf{A}(G)\boldsymbol{\rho}(G)^2/12$	$3\mathbf{I}_2(G)$	$3.5\mathbf{I}_2(G)$	
Circle	$0.9167\pi r^4$	$\underline{0.5\pi r^4}$	$0.5833\pi r^4$	$0.5\pi r^4$
Square	$0.2292\, a^4$	$0.125\, a^4$	$\underline{0.1458\, a^4}$	$0.1406\, a^4$
Equilateral triangle	$0.0331\, a^4$	$0.018\, a^4$	$\underline{0.021\, a^4}$	$0.0217\, a^4$
Ellipse with $a/b = 2$	$5.7596\, b^4$	$4.3196\, b^4$	$\underline{5.0396\, b^4}$	$5.0266\, b^4$
Ellipse with $a/b = 5$	$\underline{14.399\, b^4}$	$11.6239\, b^4$	$13.5612\, b^4$	$15.1038\, b^4$
Narrow ellipse	$\underline{0.9167\pi\, ab^3}$	$0.75\pi\, ab^3$	$0.875\pi\, ab^3$	$\pi\, ab^3$
Rectangle with $a/b = 2$	$\underline{0.4583\, b^4}$	$0.375\, b^4$	$0.4375\, b^4$	$0.4574\, b^4$
Rectangle with $a/b = 4$	$0.9167\, b^4$	$0.875\, b^4$	$\underline{1.0208\, b^4}$	$1.1232\, b^4$
Narrow rectangle	$0.2292\, ab^3$	$0.25\, ab^3$	$\underline{0.2917\, ab^3}$	$0.3333\, ab^3$
Narrow sector	$0.1146\, \gamma^3$	$0.0625\, \gamma^3$	$\underline{0.0729\, \gamma^3}$	$0.0833\, \gamma^3$
Semicircle	$0.3598\, r^4$	$0.252\, r^4$	$\underline{0.294\, r^4}$	$0.2976\, r^4$
Circular sector of angle $2\pi/3$	$\underline{0.54\, r^4}$	$0.4484\, r^4$	$0.5232\, r^4$	$0.5725\, r^4$
Circular sector of angle 2π	$0.7199\, r^4$	$0.6448\, r^4$	$\underline{0.7522\, r^4}$	$0.8781\, r^4$

Table 2

Inequality	Classes of domains		
	Simply connected domains	Convex domains	Polygonal domains circumscribed around a circle
$3\mathbf{I}_2 G \le \mathbf{P}(G)$	Conjecture 1	Conjecture 1	Valid
$2\mathbf{I}_2(G) + \pi\boldsymbol{\rho}(G)^4/6 \le \mathbf{P}(G)$	Conjecture 3	Valid	Valid
$3\mathbf{I}_2(G)/2 + \pi\boldsymbol{\rho}(G)^4/4 \le \mathbf{P}(G)$	Conjecture 2	Valid	Valid
$3\mathbf{I}_2(G) < 2\mathbf{P}(G)$	Valid	Valid	Valid
$\mathbf{A}(G)\boldsymbol{\rho}(G)^2 \le 2\mathbf{P}(G)$	Invalid	Valid	Valid

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