Examining College students' conceptual metaphors of infinity across Calculus coursework sequence

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This study examined College students' conceptual metaphors of duality in understanding and representing infinity. Data was collected using two open-ended tasks from N=188 students enrolled in one of the southwestern universities in the U.S. Data was analyzed by three independent experts using Fleiss's Kappa for inter-rater reliability (κ=.7165). Results reveal that students' experiences in traditional Calculus sequence are not supportive of the development of duality conception. On the contrary, it significantly strengthens the singularity perspective confirmed by Chi-square statistic ($\chi 2 = 9.686$, $p < .05$). Findings could serve as a facilitating instrument to further analyze cognitive obstacles in College students' understanding of the infinity concept.

Keywords: Teaching and learning of specific topics in university mathematics, Teaching and learning of analysis and calculus, Conceptual metaphors, Infinity concept.

INTRODUCTION

The concept of infinity is one of the most important, and yet challenging links in the College mathematics sequence for undergraduate science, technology, engineering and mathematics (STEM) students. Studies have confirmed that most students have extensive difficulty with the notions of infinity (Gray and Tall, 1994; Monaghan, 2001; Selden, 2002; Sfard, 1991).

Most of the prior research on College students' concept of infinity focused on the development of the concept within a particular course (Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros, & Weller, 2014; Falk, 2009; Kolar & Hodnik Čadež, 2012; Singer & Voica, 2008). This study attempts to make a contribution to the filed by closely examining the development of College students' metaphors of infinity across the entire sequence of courses, from Pre-Calculus to Calculus III.

Conceptual metaphor (Lakoff & Johnson, 1980) could be used as a way to examine students' understanding of infinity. Glucksberg (2008) posits that metaphor is a categorical assertion, whereby the attributes of one concept are

used to describe the attributes of another. Studying the development of College students' conceptual metaphors is critically important to students' success in college mathematics. Therefore, the study focused on the following *research questions*:

1. How are conceptual metaphors of infinity externalized by College students at each course in the Calculus sequence?
2. How does Calculus coursework sequence contribute to the development of College students' metaphors of infinity through the lens of duality conception?

THEORETICAL PERSPECTIVES

The study is grounded on the conceptual metaphor and blending theories (Lakoff & Johnson, 1980, Fauconnier & Turner, 2002). The study also uses mathematical idea analysis technique (Lakoff & Nunez, 2000) to examine College students' conceptual metaphors of infinity.

Nunez (2005) proposes basic mapping of infinity (BMI) as an approach to address the most challenging aspect of metaphor of infinity - its duality. Duality conception is the ability to conceive infinity as a process (potential infinity) as well as an object (actual infinity). Nunez (2005) claims that the BMI as "a double-scope conceptual blend" (p. 1729) could be used to address duality through new inferential structure integrating the finite (completed iterative process with end) and infinite (endless iterative process with no end) into blended space (an endless process with an end=actual infinity).

Aside from the BMI (Nunez, 2005), several other theories have addressed process-object duality as a model of infinity concept development: procept theory (Gray and Tall, 1994), APOS theory (Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros & Weller, 2014), reification theory (Sfard, 1991). These theories were built on Piagetian (1970, 1975) ideas of genetic epistemology, reflective abstraction, and encapsulation to conceptualize the construct of process–object duality (Gray and Tall, 1994; Sfard, 1991).

METHODOLOGY

Participants.

In order to allow for a variety of conceptions from different stages of students' learning trajectory, we selected N=188 college students enrolled in the Calculus sequence. We used cluster random sampling (Onwuegbuzie & Collins, 2007) to select participants that were willing to participate in the study 69 of which were enrolled in Pre-Calculus, 74 - in Calculus I, 26 - in Calculus II, and 19 - in

Calculus III. Majority (54%) of the participants were male and 44% were female. Three of the participant (2%) did not disclose their gender.

Data Collection.

Data was collected using the questionnaire which consisted of two open-ended tasks: (1) concept-definition task: when you think of infinity what comes to your mind; (2) concept-image task: draw infinity in the space provided. These tasks were purposefully selected to engage students in externalization of their conceptual metaphors because they relate to an individual's cognitive structure associated with the concept, and has the potential to reveal associated misconceptions of infinity that the college students may hold (Tall & Vinner, 1981).

Data Analysis.

The open-ended responses were coded by three independent expert raters using metaphor analysis and drawing methodology techniques (Cameron & Maslen, 2010; Theron, Mitchell, Smith, & Stuart, 2011). NVivo 10 software was utilized in examining students' responses to the tasks provided (table 1).

Data source	Question	Data analysis
Concept-definition task	When you think of infinity what comes to your mind?	Metaphor analysis
Concept-image task	Draw infinity in the space provided.	Visual metaphor analysis

Table 1. Data sources and data analysis

Additionally, the Delphi method was employed during the process of coding because "[t]he Delphi method is well suited as a research instrument when there is incomplete knowledge about a problem or phenomenon" (Skulmoski, Hartman & Krahn, 2007, p. 1). The number of Delphi iterations can vary from three to five (Skulmoski, Hartman & Krahn, 2007). Throughout the Delphi iterations, a consensus was reached among three raters with Fleiss's Kappa of .7165 which is considered a strong agreement (Landis & Koch, 1977).

The evolved sub-categories of *singularity and duality* along with *dominance and recessiveness* were used (Figure 1) to categorize students' conceptual metaphors of duality into the following levels:

- Level 1 - recessive singular. In this case, only one view is displayed by a student, which could either be the recessive process view or the recessive object view with a weak conviction ('p' or 'o').
- Level 2 - dominant singular. In this case, students tend to display one view in which the intensity of conviction is strong ('O' or 'P').
- Level 3 - recessive dual. This level represents the case where students demonstrate process and object views that are both recessive or one of the views is recessive (i.e., not strong or convincing). We believe this case to be an indication of emergent duality conception ('Po' or 'pO' or 'po').
- Level 4 - dominant dual. This case indicated strong and convincing duality of students' process-object view ('P' and 'O' denoted as 'PO').

Singular	Dual
Recessive	Dominant

Figure 1. Operationalization of the process-object duality framework

Finally, non-parametric technique (Chi-square statistic) was selected to measure the variance between independent groups of participants across the Calculus sequence. The selection of Chi-square was also based on the categorical nature of data used for determining students' levels of duality conception.

FINDINGS

Results of students' responses to Task 1 are displayed in Table 2.

Conception	Levels	Pre-Cal N/%	Cal 1 N/%	Cal 2 N/%	Cal 3 N/%	Total N/%
Not determinable	ND	3/5	4/5	2/8	0/0	9/5
Singularity conception	Level 1	35/51	47/63	12/48	13/69	107/57
	Level 2	28/40	16/22	4/14	6/31	54/29
Duality conception	Level 3	2/3	7/10	7/26	0/0	16/8
	Level 4	1/1	0/0	1/4	0/0	2/1

Table 2. Levels of students' duality conception in response to concept-definition task

The majority (86%) of the responses were categorized as singularity conception at Levels 1 and 2 while 9% were categorized as duality conception at Levels 3 and 4. The remaining 5% of students' responses could not be determined and were categorized as ND since either the response was blank or there was not sufficient information to determine the view. More than half of the participants (58%) in Level 1 used process metaphors to describe infinity while the remaining participants (42%) used the object metaphors. The ability to conceive of infinity as "an endless process with an end" (Nunez, 2005) such as a "*number*", an "*amount*", a "*set*", a "*cardinality*", "*something*", a "*limit*", etc. were considered as object metaphors (actual infinity) and the ability to conceive of infinity as "process with no end" (Nunez, 2005) such as "*continuing*", "*going on and on*", "*forever*", "*never ending", "over and over",* etc. were considered as process metaphors of infinity (potential infinity). Selected examples of students' conceptual metaphors are illustrated below.

Metaphor 1 - infinity as "something". Analysis using NVivo (figure 2) showed that "*something*" appeared most often (58 times) in students' responses to Task 1 across the Calculus sequence. Participants predominantly used process metaphors to describe infinity.

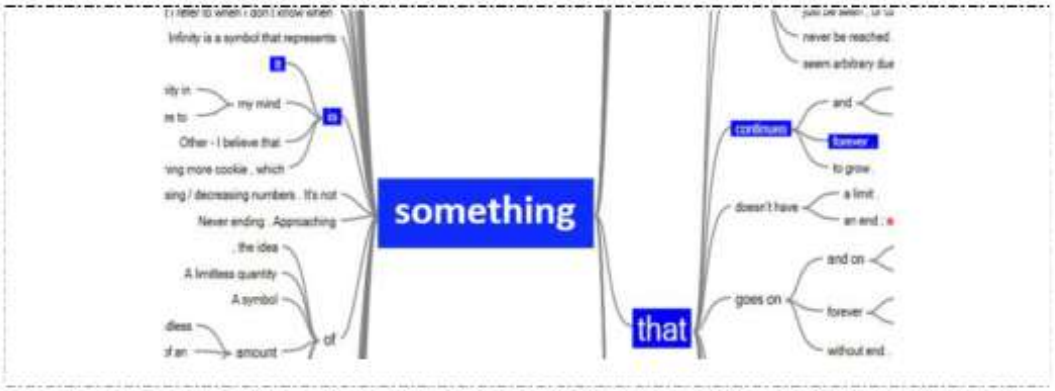

Figure 2. Example of the metaphor "something" to define infinity in Task 1

Metaphor 2 - infinity as a "number". Another most frequent metaphor in students' definition of infinity is "*number*" and it appeared 35 times. Based on the experts' rating, student's ability to conceive of infinity as "*a large number sequence*" is considered to be an indication of the object conception, however the phrase "*that does not end*" highlights the metaphor of the process (figure 3).

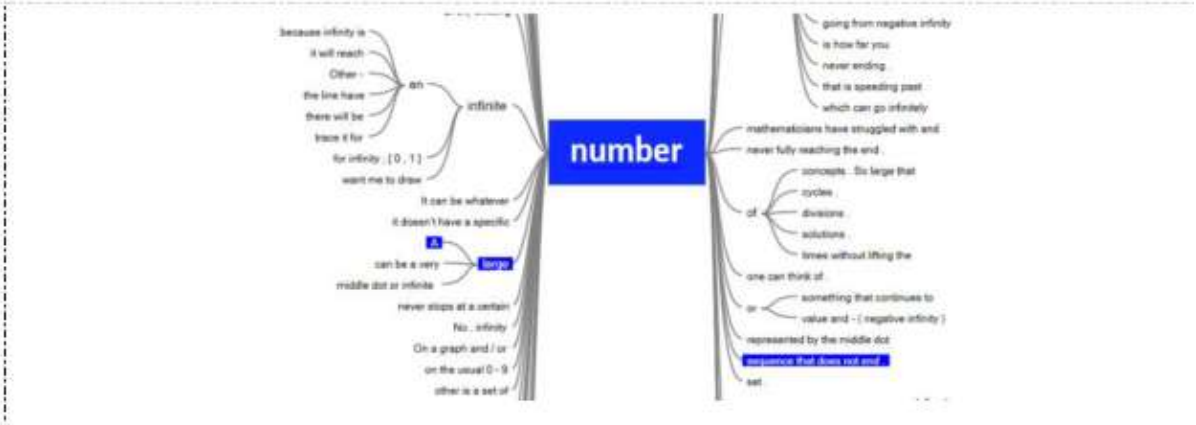

Figure 3. Example of the metaphor "number" to define infinity in Task 1

Metaphor 3 - infinity as "endless". The NVivo generated word tree in Figure 4 establishes how students used the metaphor "*endless*" which was among one the most frequent metaphor used to define infinity and it appeared 17 times.

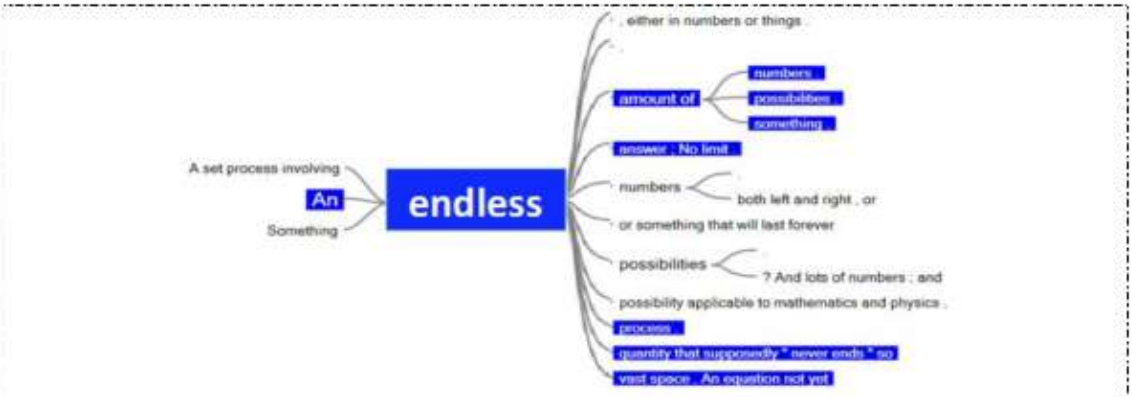

Figure 4. Examples of the metaphor "endless" to define infinity in Task 1

The second task of the questionnaire: "Draw infinity in the space provided. Explain your drawing." Table 3 aggregates students' visual metaphors of infinity including graphs, circles, arrows, number lines and blank spaces, to name a few. Some students used multiple ways to represent infinity. It was not surprising that a majority of the students (68%) drew an infinity symbol (∞) with predominately (53%) process metaphor, e.g. *"It is a symbol that has no starting point and no ending point"*.

Representation	N/%
Symbol of infinity	128/68
Graph	26/14

Number line	19/10
Blank space	11/6
Arrow	8/4
Circle	6/3
Numbers	2/1
Ellipsis	2 /1

Table 3. Students' metaphoric representations of infinity in response to Task 2

Experts' rating of the responses to Task 2 (Table 4) indicates that 5% of responses were categorized as duality conception at Levels 3 and 4. The most widely held conception of the draw infinity task is the singularity conception (63%) with a predominantly process view of infinity (71% of responses within singularity conception category). The conception of 32% participants could not be determined (ND) because either the participants did not provide an explanation to their drawings or the explanation was not sufficient to determine the participant's view. Considerably less number of students (10%) drew infinity sign as an indication of an object view.

Conception	Views	Pre-Cal	Cal 1	Cal 2	Cal 3	Total
		N/%	N/%	N/%	N/%	N/%
Not determinable	ND	18/26	26/35	11/41	4/21	59/31
Singularity conception	'p' or 'P'	34/50	27/36	12/48	13/68	86/46
	'o' or 'O'	7/10	9/12	0/0	0/0	16/8.5
Duality conception	'Po' or 'pO' or 'po'	8/11	11/15	3/11	2/11	24/13
	'PO'	2/3	1/2	0/0	0/0	3/1.5

Table 4. Results of students' responses to Task 2

Below (figure 5) we present an example of a student using object metaphor - "*an infinite number or value*". A few students (6%) used blank space to represent infinity.

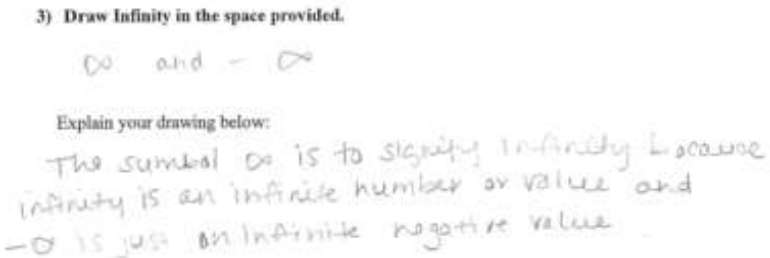

Figure 5. Example of a student's visual metaphor of infinity and its explanation

Finally, we report aggregated findings across the Calculus coursework sequence by conceptions and levels in a contingency table. Table 5 clearly illustrates that every next course in the Calculus sequence has more students possessing singularity conception: it gradually increases from Pre-Calculus (59%) to Calculus 1 (61%) and further to Calculus 2 (73%) and Calculus 3 (95%). Consequently, percentage of students having duality conception decreases from 41% in Pre-Calculus and 39% in Calculus 1 to 27% in Calculus 2 and, finally, 5% in Calculus 3. Chi-square statistic was applied to analyze the data presented in the contingency table 8. Chi-square result ($\chi^2 = 9.686$, $df = 3$, $p < .05$) confirmed that the variation in students' conceptions across the Calculus coursework sequence is significant.

Conception	Pre-Cal	Cal 1	Cal 2	Cal 3	Total
	N	N	N	N	N
Singularity	41	45	19	18	123
Duality	28	29	7	1	65
Total	69	74	26	19	188

Table 5. Contingency table of students' metaphors across Calculus sequence

Overall, this finding is an indicator of the trend observed throughout the current study that the traditional Calculus sequence promotes *a process dominated metaphor* of infinity.

DISCUSSION AND CONCLUSION

The current study contributes methodologically to the field of semiotics in mathematics education research by its ways of addressing the double-scope conceptual blend framework (Nunez, 2005) to address metaphoric process-object nature of infinity (Arnon, et al., 2014; Gray & Tall, 1994; Sfard, 1991). This study proposed the operationalized framework which allowed for granulized level of examination and interpretation of students' conceptual metaphors of infinity through *duality* lens. Furthermore, the proposed

framework enabled us to elucidate a better strategy that could guide researchers in categorizing students' metaphors of infinity into different levels to assess students' spontaneous *singularity and/or duality* conception of infinity.

This study examined the duality conception of infinity of the students enrolled at different Calculus sequence courses. In the light of the presented outcomes of this study, the following major findings addressing the research questions could be suggested: a) students' conceptual metaphors of infinity across the Calculus coursework sequence is predominantly based on the process (potential infinity) perspective; b) students' conceptual metaphors of duality is not well developed across the Calculus sequence; c) moreover, the traditional Calculus coursework sequence promotes a singularity conception of infinity. Among the major *limitations* of the study are the cluster sampling and uneven number of participants representing different courses in the Calculus sequence. The result might have been different if the same sample sizes of students in the Calculus sequence were selected and if the students were studied from Pre-Calculus through Calculus III to determine if the coursework actually supports the development of the duality conception as students progress through the Calculus sequence.

Overall, the results of our study reveal that college students' experiences in the traditional Calculus coursework sequence are not supportive of the development of a duality conception of infinity. On the contrary, the study suggests that it strengthens students' singularity process-oriented perspective on infinity. Therefore, it is critically important to provide college students with relevant experiences to build the concept of duality, which will help them further to succeed in advanced mathematical courses. Furthermore, the results of this study could serve as a facilitating instrument to analyze cognitive obstacles in college students' understanding of dual nature of infinity.

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