Siberian Mathematical Journal, Vol. 53, *No.* 2, *pp.* 207–216, 2012

DOMINATED CONVERGENCE IN MEASURE ON SEMIFINITE VON NEUMANN ALGEBRAS AND ARITHMETIC AVERAGES OF MEASURABLE OPERATORS

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Abstract: Consider a von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ. We prove that each order bounded sequence of τ-compact operators includes a subsequence whose arithmetic averages converge in the measure τ. We prove a noncommutative analog of Pratt's lemma for $L_1(\mathcal{M},\tau)$. The results are new even for the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of bounded linear operators with the canonical trace $\tau = \mathrm{tr}$ on a Hilbert space $\mathcal{H}$. We apply the main result to $L_p(\mathcal{M},\tau)$ with $0 < p \le 1$ and present some examples that show the necessity of passing to the arithmetic averages as well as the necessity of τ-compactness of the dominant.

Keywords: Hilbert space, von Neumann algebra, normal semifinite trace, measurable operator, topology of convergence in measure, spectral theorem, Banach space, Banach–Saks property, arithmetic average

Introduction

It is known (see Example 3.4 below and Theorem 2.6.7 of [1]) that a sequence of random variables $\{\xi_n\}_{n=1}^{\infty}$ on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ can vanish in probability, while the sequence of arithmetic averages

$$\left\{\frac{1}{n}\sum_{k=1}^{n}\xi_k\right\}_{n=1}^{\infty}$$

need not vanish in probability. But if $\{\xi_n\}_{n=1}^{\infty}$ are uniformly integrable then

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k \to 0$$

in probability. The law of large numbers yields the convergence in probability of the arithmetic averages of independent identically distributed integrable random variables [2, Chapter III, § 3, Theorem 2]. The existence of subsequences with converging arithmetic averages is related to the Banach–Saks property of Banach spaces. Study appeared in this context is given in [3].

Consider a von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ. Some results on the convergence of arithmetic averages of measurable operators are obtained in [4] in the framework of Segal's theory of noncommutative integration [5]. The law of large numbers for a sequence of independent identically distributed operators in $L_1(\mathcal{M},\tau)^{\mathrm{h}}$ is established in Theorem 5.4 of [6].

In this article we prove that each order bounded sequence of τ-compact operators includes a subsequence whose arithmetic averages converge in the measure τ. We obtain a noncommutative analog of Pratt's lemma for $L_1(\mathcal{M},\tau)$. The results are new even for the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ with the canonical trace $\tau = \mathrm{tr}$. We apply the main result to $L_p(\mathcal{M},\tau)$ for $0 < p \le 1$ and present some examples that show the necessity of passing to the arithmetic averages and the necessity of τ-compactness of the dominant. These results were partially announced (without proof) in the brief note [7].

The authors were supported by the Ministry for Education and Science (State Contract 02.740.11.0193).

Kazan. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 53, No. 2, pp. 258–270, March–April, 2012. Original article submitted February 25, 2011.

§ 1. Definitions and Notation

Consider a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, the projection lattice $\mathcal{M}^{\mathrm{pr}}$ of $\mathcal{M}$, and the identity element I of $\mathcal{M}$. Given $P \in \mathcal{M}^{\mathrm{pr}}$, put $P^{\perp} = I - P$. Consider a faithful normal semifinite trace τ on $\mathcal{M}$. Denote by $\|\cdot\|$ the C^*-norm on $\mathcal{M}$.

A closed operator X in $\mathcal{M}$ whose domain $\mathcal{D}(X)$ is everywhere dense in $\mathcal{H}$ is called *τ-measurable* if for every $\varepsilon > 0$ there exists $P \in \mathcal{M}^{\mathrm{pr}}$ with $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a $*$-algebra under the adjoints, multiplication by scalars, and the operations of strong addition and multiplication obtained as the closures of ordinary algebraic operations.

Given $X \in \widetilde{\mathcal{M}}$, put $|X| = \sqrt{X^*X}$. Given a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ its positive and hermitian parts respectively. Denote by $\leq$ the partial order on $\widetilde{\mathcal{M}}^{\mathrm{h}}$ generated by the proper cone $\widetilde{\mathcal{M}}^{+}$. Let $X_n \downarrow X$ stand for $X_n \leq X_m$ for $m \leq n$ and $X = \inf_n X_n$.

Endow the $*$-algebra $\widetilde{\mathcal{M}}$ with the topology t_τ of convergence in measure (see [8, 9]) whose fundamental system of neighborhoods of zero comprises the sets

$$U_{\varepsilon,\delta} = \{X \in \widetilde{\mathcal{M}} : \exists P \in \mathcal{M}^{\mathrm{pr}}\ (\|XP\| \leq \varepsilon,\ \tau(P^{\perp}) \leq \delta)\}, \quad \varepsilon > 0,\ \delta > 0.$$

It is known that $(\widetilde{\mathcal{M}}, t_\tau)$ is a complete metrizable topological $*$-algebra; moreover, $\mathcal{M}$ is dense in $(\widetilde{\mathcal{M}}, t_\tau)$. To denote the convergence of a sequence $\{X_n\}_{n=1}^{\infty} \subset \widetilde{\mathcal{M}}$ to $X \in \widetilde{\mathcal{M}}$ in t_τ we write $X_n \xrightarrow{\tau} X$; furthermore, we say that $\{X_n\}_{n=1}^{\infty}$ *converges to* X in τ or *τ-converges to* X.

Given an operator $X \in \widetilde{\mathcal{M}}$, denote by $\mu_t(X)$ its *nonincreasing rearrangement*, which is the function $\mu(X) : (0, \infty) \to [0, \infty)$, defined as

$$\mu_t(X) = \inf\{\|XP\| : P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^{\perp}) \leq t\}, \quad t > 0.$$

The topology t_τ is determined by the F-norm

$$\rho_\tau(X) = \inf_{t>0} \max\{t, \mu_t(X)\}, \quad X \in \widetilde{\mathcal{M}}.$$

The set of τ-compact operators

$$\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \lim_{t\to\infty} \mu_t(X) = 0\}$$

is an ideal in $\widetilde{\mathcal{M}}$.

Denote by m the Lebesgue measure on $\mathbb{R}$. We can define the noncommutative L_p-Lebesgue space $(0 < p < \infty)$ associated with $(\mathcal{M}, \tau)$ as

$$L_p(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}} : \mu(X) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm (the norm for $1 \leq p < \infty$) $\|X\|_p = \|\mu(X)\|_p$ for $X \in L_p(\mathcal{M}, \tau)$. Denote the extension of τ to a unique linear functional on $\mathcal{M} \cap L_1(\mathcal{M}, \tau)$, and so on the whole of $L_1(\mathcal{M}, \tau)$, by the same letter τ.

A Banach space $\mathcal{E}$ enjoys the *Banach–Saks property* (see [10] for instance) if from every bounded sequence $\{X_n\}_{n=1}^{\infty}$ in $\mathcal{E}$ we can refine a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ whose arithmetic averages $\frac{1}{k}\sum_{i=1}^{k} X_{n_i}$ converge in norm.

Every uniformly convex Banach space enjoys the Banach–Saks property [11]. For $1 < p < \infty$ the space $L_p(\mathcal{M}, \tau)$ is uniformly convex [12]. The continuity of operator functions on $(\widetilde{\mathcal{M}}, t_\tau)$ is studied in [13–15].

Remark 1.1. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$ then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$ and the topology t_τ coincides with the topology of the norm $\|\cdot\|$. Furthermore, $\widetilde{\mathcal{M}}_0$ is the ideal of compact operators on $\mathcal{H}$ and

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of singular values of X, which are the eigenvalues of the operator $|X|$ taken in decreasing order of multiplicity, and χ_A stands for the indicator of a set $A \subset \mathbb{R}$. For $0 < p < \infty$ we have $L_p(\mathcal{B}(\mathcal{H}), \mathrm{tr}) = \mathfrak{S}_p(\mathcal{H})$, the Schatten–von Neumann ideal.

If $\mathcal{M}$ is abelian then $\mathcal{M} \simeq L^{\infty}(\Omega, \Sigma, \nu)$ and $\tau(f) = \int_{\Omega} f d\nu$, where (Ω, Σ, ν) is a localizable measure space, and $\widetilde{\mathcal{M}}$ coincides with the algebra of all measurable complex functions f on (Ω, Σ, ν) which are bounded outside a set of finite measure. Furthermore, t_τ is the usual topology of convergence in measure. The rearrangement $\mu_t(f)$ coincides with the nonincreasing rearrangement of the function $|f|$.

If $\tau(I) < \infty$ then $\widetilde{\mathcal{M}}$ consists of all closed linear operators on $\mathcal{H}$ associated to $\mathcal{M}$. Furthermore, t_τ is independent of a concrete choice of a trace and is minimal among all metrizable topologies which agree with the ring structure on $\widetilde{\mathcal{M}}$ (see [16]).

§ 2. Dominated τ-Convergence

Lemma 2.1. *If $X_k \in \widetilde{\mathcal{M}}$ and $\lambda_k > 0$ for $k = \overline{1, n}$ with $\sum_{k=1}^{n} \lambda_k \leq 1$ then*

$$\left|\sum_{k=1}^{n} \lambda_k X_k\right|^2 \leq \sum_{k=1}^{n} \lambda_k |X_k|^2. \tag{1}$$

If $Y \in \widetilde{\mathcal{M}}^+$ and $|X_k|^2 \leq Y$ for $k = \overline{1, n}$ then

$$\left|\sum_{k=1}^{n} \lambda_k X_k\right| \leq \sqrt{Y}. \tag{2}$$

Proof. For $n = 1$ the first inequality is obvious. The inequality $(X_1 - X_2)^*(X_1 - X_2) \geq 0$ yields $X_1^* X_2 + X_2 X_1^* \leq X_1^* X_1 + X_2^* X_2$. Use the method of mathematical induction. For $n = 2$ we have

$$\begin{aligned}|\lambda_1 X_1 + \lambda_2 X_2|^2 &= \lambda_1^2 X_1^* X_1 + \lambda_1 \lambda_2 (X_1^* X_2 + X_2^* X_1) + \lambda_2^2 X_2^* X_2 \\ &\leq (\lambda_1 + \lambda_2)(\lambda_1 X_1^* X_1 + \lambda_2 X_2^* X_2) \leq \lambda_1 |X_1|^2 + \lambda_2 |X_2|^2.\end{aligned}$$

Suppose that (1) is satisfied for all $X_k \in \widetilde{\mathcal{M}}$ and $\lambda_k > 0$ for $k = \overline{1, n-1}$ with $\sum_{k=1}^{n-1} \lambda_k \leq 1$. Put

$$t_k = \frac{\lambda_k}{\sum\limits_{k=1}^{n-1} \lambda_k}, \quad k = \overline{1, n-1}.$$

Then $\sum_{k=1}^{n-1} t_k = 1$ and the inductive assumption yields

$$\left|\sum_{k=1}^{n-1} t_k X_k\right|^2 \leq \sum_{k=1}^{n-1} t_k |X_k|^2. \tag{3}$$

By (1) and (3) in the case $n = 2$, we have

$$\left|\sum_{k=1}^{n} \lambda_k X_k\right|^2 = \left|\left(\sum_{k=1}^{n-1} \lambda_k X_k\right) + \lambda_n X_n\right|^2 \leq \left(\sum_{k=1}^{n-1} \lambda_k |X_k|^2\right) + \lambda_n |X_n|^2 \leq \sum_{k=1}^{n} \lambda_k |X_k|^2.$$

Since $t \mapsto \sqrt{t}$ for $0 \leq t < \infty$ is an operator monotone function, (1) implies that

$$\left|\sum_{k=1}^{n} \lambda_k X_k\right| \leq \left(\sum_{k=1}^{n} \lambda_k |X_k|^2\right)^{1/2}. \tag{4}$$

Furthermore, if $|X_k|^2 \leq Y \in \widetilde{\mathcal{M}}^+$ for $k = \overline{1, n}$ then, using once more the operator monotonicity of $t \mapsto \sqrt{t}$ for $0 \leq t < \infty$, we deduce (2) from (4). $\square$

In particular, Lemma 2.1 yields

$$\frac{1}{n} \sum_{k=1}^{n} |X_k|^2 \geq \left|\frac{1}{n} \sum_{k=1}^{n} X_k\right|^2.$$

Theorem 2.2. *Consider a von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ. Given $X_n, X \in \widetilde{\mathcal{M}}_0$ with $X \geq 0$ and $|X_n|^2 \leq X$ for $n \in \mathbb{N}$, there exists a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ whose arithmetic averages*

$$\widetilde{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_{n_i} \tag{5}$$

t_τ-converge to some operator $\widetilde{X} \in \widetilde{\mathcal{M}}_0$ with $|\widetilde{X}|^2 \leq X$.

Proof. Fix some $\varepsilon > 0$. Take a spectral projection $P \in \mathcal{M}^{\mathrm{pr}}$ of X such that $\tau(P) < \infty$ and $\|XP^{\perp}\| \leq \varepsilon^2$. Put $Y_n \equiv X_n(I+X)^{-1/2}P$. Then

$$\begin{aligned}|Y_n|^2 &= P(I+X)^{-1/2}X_n^* X_n(I+X)^{-1/2}P \\ &\leq P(I+X)^{-1/2}X(I+X)^{-1/2}P \leq PIP = P, \quad n \in \mathbb{N}.\end{aligned}$$

Given a bounded sequence $\{Y_n\}_{n=1}^{\infty}$ in the Hilbert space $L_2(\mathcal{M},\tau)$, by the Banach–Saks property there exist an operator $Y \in L_2(\mathcal{M},\tau)$ and a subsequence $\{Y_{n_i}\}_{i=1}^{\infty}$ with

$$\frac{1}{k}\sum_{i=1}^{k} Y_{n_i} \equiv \widetilde{Y}_k \to Y \quad \text{as } k \to \infty \text{ in } L_2(\mathcal{M},\tau).$$

Since the natural embedding of $L_2(\mathcal{M},\tau)$ into $(\widetilde{\mathcal{M}}_0, t_\tau)$ (see [17, Theorem 3.2] or [16, Theorem 1] for instance) is continuous, we have $\widetilde{Y}_k \xrightarrow{\tau} Y$ as $k \to \infty$. Thus, $\widetilde{Y}_k(I+X)^{1/2} \xrightarrow{\tau} Y(I+X)^{1/2}$ as $k \to \infty$ since the multiplication in $\widetilde{\mathcal{M}}$ is t_τ-continuous.

We showed already that $\widetilde{X}_k P \xrightarrow{\tau} Y(I+X)^{1/2}$ as $k \to \infty$. Therefore, the sequence $\{\widetilde{X}_k P\}_{k=1}^{\infty}$ is t_τ-fundamental:

$$\forall \varepsilon > 0\, \exists N = N(\varepsilon)\, \forall k, m \geq N \quad (\widetilde{X}_k P - \widetilde{X}_m P \in U_{\varepsilon,\varepsilon}). \tag{6}$$

Since Lemma 2.1 implies that

$$|\widetilde{X}_k|^2 \leq \sum_{i=1}^{k} \frac{1}{k}|X_{n_i}|^2 \leq \frac{1}{k}\sum_{i=1}^{k} X = X, \quad k \in \mathbb{N}, \tag{7}$$

we have

$$|\widetilde{X}_k P^{\perp}|^2 = P^{\perp}|\widetilde{X}_k|^2 P^{\perp} \leq P^{\perp}XP^{\perp} \leq \varepsilon^2 P^{\perp}$$

and the operator monotonicity of $\lambda \mapsto \sqrt{\lambda}$ for $\lambda \geq 0$ yields $|\widetilde{X}_k P^{\perp}| \leq \varepsilon P^{\perp}$, $k \in \mathbb{N}$. Consequently, $\widetilde{X}_k P^{\perp} \in U_{\varepsilon,\varepsilon}$, $k \in \mathbb{N}$. Since

$$U_{a,b} + U_{c,d} \subset U_{a+c,b+d} \tag{8}$$

for all $a, b, c, d > 0$ (see [8, 9]), we have

$$\forall k, m \in \mathbb{N} \quad (\widetilde{X}_k P^{\perp} - \widetilde{X}_m P^{\perp} \in U_{2\varepsilon,2\varepsilon}). \tag{9}$$

Taking (8) into account, we infer from (6) and (9) that

$$\forall \varepsilon > 0\, \exists N = N(\varepsilon)\, \forall k, m \geq N \quad (\widetilde{X}_k - \widetilde{X}_m \in U_{3\varepsilon,3\varepsilon}).$$

Therefore, the sequence $\{\widetilde{X}_k\}_{k=1}^{\infty}$ is t_τ-fundamental. Since the ideal $\widetilde{\mathcal{M}}_0$ is t_τ-closed, this sequence t_τ-converges to some operator $\widetilde{X} \in \widetilde{\mathcal{M}}_0$. We have $\widetilde{X}_k^* \xrightarrow{\tau} \widetilde{X}^*$ as $k \to \infty$ since the involution in $\widetilde{\mathcal{M}}$ is t_τ-continuous. Since the product of operators is t_τ-continuous on $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$, we obtain $|\widetilde{X}_k|^2 = \widetilde{X}_k^* \widetilde{X}_k \xrightarrow{\tau} \widetilde{X}^* \widetilde{X} = |\widetilde{X}|^2$ as $k \to \infty$. Since $X - |\widetilde{X}_k|^2 \xrightarrow{\tau} X - |\widetilde{X}|^2$ as $k \to \infty$ and the cone $\widetilde{\mathcal{M}}^+$ is t_τ-closed, it follows that $|\widetilde{X}|^2 \leq X$. $\square$

An interesting case of the coincidence of t_τ with the strong operator topology on the bounded parts of $\mathcal{M}$ for $\tau(I) < \infty$ is studied in Lemma 3.1 of [18] under whose hypotheses Theorem 2.2 implies

Corollary 2.3. *Consider a von Neumann algebra $\mathcal{M}$ with a faithful normal finite trace τ. Given $X_n, X \in \mathcal{M} \cap \widetilde{\mathcal{M}}_0$ with $X \geq 0$ and $|X_n|^2 \leq X$ for $n \in \mathbb{N}$, there exists a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ whose arithmetic averages* (5) *converge in the strong operator topology to some operator $\widetilde{X} \in \mathcal{M} \cap \widetilde{\mathcal{M}}_0$ with $|\widetilde{X}|^2 \leq X$.*

REMARK 2.4. Consider the symmetric space $\mathcal{E}$ on $(\mathbb{R}^+, m)$ with the Fatou property and let

$$\mathcal{E}(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}} : \mu(X) \in \mathcal{E}\}, \quad \|X\|_{\mathcal{E}(\mathcal{M},\tau)} := \|\mu(X)\|_{\mathcal{E}}.$$

It is shown in [19, Theorem 2.8] that, given a sequence $\{Y_n\}_{n=1}^{\infty} \subset \mathcal{E}(\mathcal{M}, \tau)$ with $\sup \|Y_n\|_{\mathcal{E}(\mathcal{M},\tau)} < \infty$, there exist $Y \in \mathcal{E}(\mathcal{M}, \tau)$ and a subsequence $\{X_n\}_{n=1}^{\infty} \subseteq \{Y_n\}_{n=1}^{\infty}$ such that for every finer subsequence $\{X_{n_i}\}_{i=1}^{\infty} \subseteq \{X_n\}_{n=1}^{\infty}$ the arithmetic averages (5) t_τ-converge to Y.

This remarkable and deep result fails to cover our Theorem 2.2: if $\tau(I) < \infty$ then $\widetilde{\mathcal{M}}_0 = \widetilde{\mathcal{M}}$, while $\mathcal{E}(\mathcal{M}, \tau) \subseteq L_1(\mathcal{M}, \tau)$. Consider $\mathcal{M} = L_\infty([0,1], m)$, the function

$$X(t) = \begin{cases} t^{-2}, & \text{if } 0 < t \leq 1, \\ 0, & \text{if } t = 0, \end{cases}$$

and the sequence $X_n = \frac{n}{n+1} X^{1/2}$, $n \in \mathbb{N}$. Theorem 2.2 applies in this situation, but Theorem 2.8 of [19] is not applicable.

In the hypotheses of Corollary 2.3 we assume that $\tau(I) < \infty$ implies the inclusion $\mathcal{M} \subseteq \widetilde{\mathcal{M}}_0$. We can express these hypotheses as $X_n, X \in \mathcal{M} \subseteq L_1(\mathcal{M}, \tau)$; thus, the claim of Corollary 2.3 follows also from Theorem 2.8 of [19] and Lemma 3.1 of [18].

Below we need

Proposition 2.5 [20, Theorem 3.6]. *Take $T_n, T \in \widetilde{\mathcal{M}}$ for $n \in \mathbb{N}$ with $T_n \xrightarrow{\tau} T$. Suppose that $0 < p < \infty$ and take $S_n, S \in L_p(\mathcal{M}, \tau)$ for $n \in \mathbb{N}$ satisfying the conditions:*

(i) $\mu_t(T_n) \leq \mu_t(S_n)$ *(this holds provided that $|T_n| \leq |S_n|$);*

(ii) $\|S\|_p = \lim_{n\to\infty} \|S_n\|_p$;

(iii) $\mu_t(S) \leq \liminf_{n\to\infty} \mu_t(S_n)$ *(this holds provided that $S_n \xrightarrow{\tau} S$).*

Then $T_n, T \in L_p(\mathcal{M}, \tau)$ and $\lim_{n\to\infty} \|T - T_n\|_p = 0$. If $p = 1$ then $\lim_{n\to\infty} \tau(T_n) = \tau(T)$.

Corollary 2.6. *Suppose that $0 < p < \infty$. Consider a von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ. Given $X_n, X \in L_p(\mathcal{M}, \tau)$ with $X \geq 0$ and $|X_n|^2 \leq X^2$ for $n \in \mathbb{N}$, there exists a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ whose arithmetic averages* (5) *converge in $L_p(\mathcal{M}, \tau)$ to some operator $\widetilde{X}$ with $|\widetilde{X}|^2 \leq X^2$.*

PROOF. For $1 < p < \infty$ the claim of the corollary follows from the Banach–Saks property of $L_p(\mathcal{M}, \tau)$. Suppose that $0 < p \leq 1$. By Theorem 2.2, there exists a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ whose arithmetic averages (5) satisfy $|\widetilde{X}_n|^2 \leq X^2$ for $n \in \mathbb{N}$ (see (7)) and which t_τ-converges to some operator $\widetilde{X} \in \widetilde{\mathcal{M}}_0$ with $|\widetilde{X}|^2 \leq X^2$. Thus, $|\widetilde{X}_n| \leq X$ for $n \in \mathbb{N}$ and $|\widetilde{X}| \leq X$ since the function $\lambda \mapsto \sqrt{\lambda}$ for $\lambda \geq 0$ is operator monotone and $\widetilde{X} \in L_p(\mathcal{M}, \tau)$. It remains to apply Proposition 2.5 with $T_n = \widetilde{X}_n$ and $S_n = X$ for $n \in \mathbb{N}$. $\square$

Observe that Corollary 2.6 for $0 < p \leq 1$ coincides with Theorem 2.3 of [4], but the proof here is new.

Proposition 2.7. *Consider a von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ. Take $X_n, X \in \widetilde{\mathcal{M}}_0^h$ satisfying*

$$X_1 \leq X_2 \leq \cdots \leq X_n \leq \cdots \leq X, \quad n \in \mathbb{N},$$

and put $Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k$. Then there exists $Y \in \widetilde{\mathcal{M}}_0^h$ such that $X_n \xrightarrow{\tau} Y$ and $Y_n \xrightarrow{\tau} Y$ as $n \to \infty$.

PROOF. Observe that

$$X \geq X_n \geq Y_n = Y_{n-1} + \frac{1}{n^2 - n} \sum_{k=1}^{n-1} (X_n - X_k) \geq Y_{n-1}, \quad n \geq 2.$$

Without loss of generality we may assume that $X_1 \geq 0$; otherwise, put $\widetilde{X_n} = X_n + |X_1|$, and then $\widetilde{Y_n} = Y_n + |X_1|$ for $n \in \mathbb{N}$. We have

$$Y_{n^2} \geq \frac{X_{n+1} + \cdots + X_{n^2}}{n^2} \geq \frac{n^2 - n}{n^2} X_{n+1}, \quad n \in \mathbb{N}.$$

By Proposition 1.1 of [21], the sharp upper bound $Y = \sup_n X_n = \sup_n Y_n \in \widetilde{\mathcal{M}}^+$ exists. Since $0 \leq Y \leq X$, we can find an operator $Z \in \mathcal{M}$ with $\|Z\| \leq 1$ and $Y = ZXZ^*$ (see [22, the Proposition on p. 261]). Since $\widetilde{\mathcal{M}}_0$ is an ideal of $\widetilde{\mathcal{M}}$, we have $Y \in \widetilde{\mathcal{M}}_0^+$. Consequently, $Y - X_n \in \widetilde{\mathcal{M}}_0^+$ and $Y - X_n \downarrow 0$ (as $n \to \infty$). Thus, $Y - X_n \xrightarrow{\tau} 0$ as $n \to \infty$ by Lemma 3.14 of [23]. Similarly, $Y_n \xrightarrow{\tau} Y$ as $n \to \infty$.

The conditions of monotonicity and boundedness of the sequence $\{X_n\}_{n=1}^{\infty}$ are essential (see Example 3.4(b) below). $\square$

Remark 2.8. Consider an F-normed space $\mathcal{E}$ and take $X_n \in \mathcal{E}$ for $n \in \mathbb{N}$. Put $Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. We have

$$\frac{1}{n} X_n = Y_n - \frac{n-1}{n} Y_{n-1}, \quad n \geq 2.$$

If $\{Y_n\}_{n=1}^{\infty}$ is fundamental in $\mathcal{E}$ then $\frac{1}{n} X_n \to 0$ in $\mathcal{E}$ as $n \to \infty$.

§ 3. Examples

3.1. (a) The condition $X \in \widetilde{\mathcal{M}}_0$ in Theorem 2.2 is essential.

Take $\Omega = (0, \infty)$, the Lebesgue measure m on Ω, and the von Neumann algebra $\mathcal{M} = \widehat{L}_\infty(\Omega, m)$ acting (by multipliers) on the Hilbert space $L_2(\Omega, m)$. Put $X = \chi_\Omega$ and $X_n = \chi_{(0,n]}$. Then $X_n \leq X$ for $n \in \mathbb{N}$. Observe that $X_n \in \widetilde{\mathcal{M}}_0$ $(n \in \mathbb{N})$, but $X \notin \widetilde{\mathcal{M}}_0$. Every subsequence $1 \leq n_1 < n_2 < \cdots < n_k < \cdots$ of $\mathbb{N}$ satisfies $n_k \geq k$ for all $k \in \mathbb{N}$. Put $\widetilde{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_{n_i}$ and $Z_k = \widetilde{X}_{2k} - \widetilde{X}_k$ for $k \in \mathbb{N}$. We have

$$\widetilde{X}_k(\omega) = \begin{cases} 1, & \text{if } 0 < \omega < n_1, \\ 1 - \frac{m}{k}, & \text{if } n_m < \omega \leq n_{m+1},\ m = \overline{1, k-1}, \\ 0, & \text{if } \omega > n_k; \end{cases}$$

$$Z_k(\omega) = \begin{cases} 0, & \text{if } 0 < \omega < n_1 \text{ or } \omega > n_{2k}, \\ \frac{m}{2k}, & \text{if } n_m < \omega \leq n_{m+1},\ m = \overline{1, k-1}, \\ 1 - \frac{m}{2k}, & \text{if } n_m < \omega \leq n_{m+1},\ m = \overline{k, 2k-1}. \end{cases}$$

Consequently, $Z_k(\omega) = 1/2$ if $n_k < \omega \leq n_{k+1}$; therefore, the sequence $\{\widetilde{X}_k\}_{k=1}^{\infty}$ is not m-fundamental.

(b) In Theorem 2.2 we cannot weaken the condition $|X_n|^2 \leq X$ for $n \in \mathbb{N}$ to $\mu(X_n)^2 \leq \mu(X)$ for $n \in \mathbb{N}$. In the framework of Example (a) consider the sequence

$$X_n(\omega) = \begin{cases} 0, & \text{if } 0 < \omega \leq n-1 \text{ or } \omega > n, \\ (\omega - n + 1)^{-1}, & \text{if } n - 1 < \omega \leq n. \end{cases}$$

Then $X_n \in \widetilde{\mathcal{M}}_0$ and $\mu(X_n) = X_1$ for all $n \in \mathbb{N}$. Since the supports of the functions X_n are disjoint, passage to a subsequence changes nothing. For $Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ we have $m\{\omega \in \Omega : Y_n(\omega) \geq 1\} = 1$. Since $|Y_{2n} - Y_n| = Y_{2n}$ for all $n \in \mathbb{N}$, the sequence $\{Y_n\}_{n=1}^{\infty}$ is not m-fundamental.

3.2. Examples in which passage to arithmetic averages in Theorem 2.2 is necessary.

(a) Denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathcal{H}$ and take an orthonormal system $\{\xi_k\}_{k=1}^{\infty}$. Consider in $\mathcal{B}(\mathcal{H})$ the sequence of partial isometries $A_n = \langle \cdot, \xi_1 \rangle \xi_n$. Then $A_n^* = \langle \cdot, \xi_n \rangle \xi_1$ and $A_n A_n^* = \langle \cdot, \xi_n \rangle \xi_n = P_n \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ for $n \in \mathbb{N}$. Since $A_n^* A_n = P_1$, it follows that $|A_n| = P_1$ $(n \in \mathbb{N})$. The sequence $\{A_n\}_{n=1}^{\infty}$ lacks converging subsequences:

$$\|A_n - A_k\| = \sup_{\|\eta\|_{\mathcal{H}} \leq 1} \|\langle \eta, \xi_1 \rangle \xi_n - \langle \eta, \xi_1 \rangle \xi_k\|_{\mathcal{H}} = \|\xi_n - \xi_k\|_{\mathcal{H}} = \sqrt{2}, \quad n \neq k.$$

We have $\left\| \frac{1}{n}\sum_{k=1}^{n} A_k \right\| = n^{-1/2} \to 0$ as $n \to \infty$.

(b) In the algebra $\mathcal{M} = L_\infty([0,1], m)$, where m is the Lebesgue measure on $[0,1]$, consider the sequence of Rademacher functions $r_n(t) = \operatorname{sign} \sin 2^n \pi t$ with $0 \le t \le 1$. It is obvious that $|r_n(t)| \le 1$ for $0 \le t \le 1$.

The sequence $\{r_n\}_{n=1}^\infty$ lacks m-converging subsequences

$$m\{t \in [0,1] : |r_n(t) - r_k(t)| \ge 1\} = m\{t \in [0,1] : r_n(t) \ne r_k(t)\} = \frac{1}{2}, \quad n \ne k.$$

Passage to a subsequence in Theorem 2.2 is not needed here: according to Khintchine's inequality [24], if $0 < p < \infty$ and $S_n(t) = \sum_{k=1}^n a_k r_k(t)$ is a polynomial with respect to the Rademacher system with real coefficients $a_1, a_2, \dots, a_n$ then

$$\int_0^1 |S_n(t)|^p \, dt \le \left(\frac{p}{2} + 1\right)^{p/2} \left(\sum_{k=1}^n a_k^2\right)^{p/2},$$

and for $a_k = 1/n$ we have $S_n(t) = \frac{1}{n}\sum_{k=1}^n r_k(t)$ and $S_n \to 0$ as $n \to \infty$ in $L_p([0,1], m)$.

3.3. Some examples in which passage to arithmetic averages in Theorem 2.2 is not obligatory.

(a) Take $\mathcal{M} = l_\infty$ and $\tau(X) = \sum_{k=1}^\infty x_k$ for $X = \{x_k\}_{k=1}^\infty \in \mathcal{M}^+$. Then $\widetilde{\mathcal{M}}_0 = c_0$ is the space of complex sequences converging to zero.

It is well known that given $X_n, X \in c_0$ and $|X_n| \le X$ for $n \in \mathbb{N}$, there exists a subsequence $\{X_{n_i}\}_{i=1}^\infty$ and $Y \in c_0$ such that $X_{n_i} \to Y$ in $\mathcal{M}$.

(b) Denote by l_p, with $0 < p < \infty$, the space of complex sequences $X = \{x_k\}_{k=1}^\infty$ satisfying the condition $\sum_{k=1}^\infty |x_k|^p < \infty$. Given $X_n, X \in l_p$ with $|X_n| \le X$ for $n \in \mathbb{N}$, there exist a subsequence $\{X_{n_i}\}_{i=1}^\infty$ and $Y \in l_p$ such that $X_{n_i} \to Y$ in l_p.

Indeed, the space l_p is embedded into c_0, and so the sequence $\{X_n\}_{n=1}^\infty$ satisfies the conditions of (a) and Proposition 2.5 with $T_i = X_{k_{n_i}}$, $T = Y$, and $S_i = S = X$ for $i \in \mathbb{N}$.

3.4. (a) An example of a sequence, vanishing in measure, whose arithmetic averages fail to converge in measure. In the algebra $\mathcal{M} = L_\infty([0,1], m)$ take

$$f_1 = f_2 = \chi_{[0,1]}, \quad f_3 = 2\chi_{[0,1/2)}, \quad f_4 = 2\chi_{[1/2,1]},$$

$$f_5 = 2^2\chi_{[0,1/2^2)}, \quad f_6 = 2^2\chi_{[1/2^2,1/2)}, \quad f_7 = 2^2\chi_{[1/2,3/2^2)}, \quad f_8 = 2^2\chi_{[3/2^2,1]},$$

$$f_9 = 2^3\chi_{[0,1/2^3)}, \quad f_{10} = 2^3\chi_{[1/2^3,1/2^2)}, \quad f_{11} = 2^3\chi_{[1/2^2,3/2^3)},$$

$$f_{12} = 2^3\chi_{[3/2^3,1/2)}, \quad f_{13} = 2^3\chi_{[1/2,5/2^3)}, \dots.$$

Then $f_n \to 0$ in the measure m, but $g_n = \frac{1}{n}\sum_{k=1}^n f_k$ fail to vanish in the measure m since $g_{2^n} = \chi_{[0,1]}$ and $g_{2^n+2^{n-1}} = \frac{2}{3}\chi_{[0,1]} + \frac{2}{3}\chi_{[0,1/2)}$ for all $n \in \mathbb{N}$.

(b) An example of a sequence, vanishing in measure, whose arithmetic averages converge in measure to the identity. In the algebra $\mathcal{M} = L_\infty([0,1], m)$ take

$$f_1 = \chi_{[0,1]}, \quad f_2 = 2\chi_{[0,1/2)}, \quad f_3 = 2\chi_{[1/2,1]}, \quad f_4 = 3\chi_{[0,1/3)}, \quad f_5 = 3\chi_{[1/3,2/3)},$$

$$f_6 = 3\chi_{[2/3,1]}, \quad f_7 = 4\chi_{[0,1/4)}, \quad f_8 = 4\chi_{[1/4,1/2)}, \quad f_9 = 4\chi_{[1/2,3/4)},$$

$$f_{10} = 4\chi_{[3/4,1]}, \quad f_{11} = 5\chi_{[0,1/5)}, \dots.$$

Then $f_n \to 0$ in the measure m, but

$$g_n = \frac{1}{n}\sum_{k=1}^n f_k \to \chi_{[0,1]}$$

uniformly on $[0,1]$: we have $1+2+\cdots+n=n(n+1)/2=k_n$ and $g_{k_n}=\chi_{[0,1]}$; if $k_n<k<k_{n+1}$ then

$$g_k=\frac{k_n}{k}\chi_{[0,1]}+\frac{n+1}{k}\chi_{[0,(k-k_n/(n+1))}$$

and

$$\frac{k_n}{k_{n+1}}<\frac{k_n}{k}<1,\quad \frac{n+1}{k}<\frac{2}{n},\quad \frac{k_n}{k_{n+1}}=\frac{n+1}{n+2}\to 1$$

as $n\to\infty$.

(c) The convergence of arithmetic averages need not be preserved in the passage to subsequences: in the sequence of (b), choose as the required subsequence the sequence of (a) starting with f_2.

§ 4. Arithmetic Averages of Measurable Operators

Lemma 4.1. *Consider a vector space $\mathcal{E}$ over the field $\mathbb{R}$ or $\mathbb{C}$. The algebraic sum of the deviations of the individual terms of a tuple $X_1,\dots,X_n\in\mathcal{E}$ from the arithmetic average $A=\frac{1}{n}\sum_{k=1}^n X_k$ vanishes:*

$$\sum_{k=1}^n (X_k-A)=0. \tag{10}$$

Proof. We have

$$\sum_{k=1}^n (X_k-A)=\sum_{k=1}^n\left(X_k-\frac{1}{n}\sum_{i=1}^n X_i\right)=\sum_{k=1}^n X_k-n\frac{1}{n}\sum_{i=1}^n X_i=0. \quad \square$$

This feature is characteristic of the arithmetic average: the latter is a unique root of the equation $\sum_{k=1}^n (X_k-X)=0$. Indeed, $\sum_{k=1}^n X_k-nX=0$, and so $X=A$.

Theorem 4.2. *The sum of squares of the absolute values of the deviations of the individual terms of the tuple $X_1,\dots,X_n\in\widetilde{\mathcal{M}}$ from the arithmetic average $A=\frac{1}{n}\sum_{k=1}^n X_k$ is less than the sum of squares of the absolute values of their deviations from each operator $B\in\widetilde{\mathcal{M}}$ but A.*

Proof. Given $S,T\in\widetilde{\mathcal{M}}^{\mathrm{h}}$, we say "$S$ is less than T" meaning that $S\le T$ and $S\ne T$. For every k we have $X_k-B=(X_k-A)-(A-B)$. Thus,

$$|X_k-B|^2=|X_k-A|^2+|A-B|^2-(X_k-A)^*(A-B)-(A-B)^*(X_k-A).$$

Summing these equalities over all k, we obtain

$$\sum_{k=1}^n |X_k-B|^2=\sum_{k=1}^n |X_k-A|^2+n|A-B|^2$$
$$-\left(\sum_{k=1}^n (X_k-A)\right)^*(A-B)-(A-B)^*\sum_{k=1}^n (X_k-A).$$

By (10),

$$\sum_{k=1}^n |X_k-B|^2=\sum_{k=1}^n |X_k-A|^2+n|A-B|^2. \quad \square \tag{11}$$

Proposition 4.3. *Consider a unitary space $\mathcal{K}$ over the field $\mathbb{R}$ or $\mathbb{C}$ with the norm $\|\cdot\|_{\mathcal{K}}$. Given $X_1,\dots,X_n\in\mathcal{K}$, we have*

$$\inf_{B\in\mathcal{K}}\sum_{k=1}^n \|X_k-B\|_{\mathcal{K}}^2=\sum_{k=1}^n \|X_k-A\|_{\mathcal{K}}^2,\quad \textit{where } A=\frac{1}{n}\sum_{k=1}^n X_k.$$

For $\mathcal{K}=L_2(\mathcal{M},\tau)$ this well-known statement also follows from (11).

Proposition 4.4. *Given* $X \in \widetilde{\mathcal{M}}^+$ *and* $n \in \mathbb{N}$, *we have*

$$\frac{I + X + X^2 + \cdots + X^{n-1}}{n} \geq X^{(n-1)/2}.$$

PROOF. Given $x > 0$ and $n \in \mathbb{N}$, the inequality between the arithmetic and geometric averages yields

$$\frac{1 + x + x^2 + \cdots + x^{n-1}}{n} \geq \sqrt[n]{1 \cdot x \cdot x^2 \cdot \ldots \cdot x^{n-1}} = \sqrt[n]{x^{n(n-1)/2}} = x^{(n-1)/2}.$$

Then we apply the spectral theorem for the selfadjoint operator X. □

REMARK 4.5. The claims of Lemma 2.1, Theorem 4.2, and Proposition 4.4 translate, with similar proofs, to the algebra $S(\mathcal{M})$ of locally measurable operators [22] associated to an arbitrary von Neumann algebra $\mathcal{M}$.

§ 5. A Noncommutative Analog of Pratt's Lemma

For Pratt's lemma for random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ see [2, pp. 227–228] for instance.

Theorem 5.1. *Consider a von Neumann algebra* $\mathcal{M}$ *with a faithful normal semifinite trace* τ *and take* $X, Z, X_n, Z_n \in \widetilde{\mathcal{M}}^{\mathrm{h}} \cap L_1(\mathcal{M}, \tau)$, *and* $Y, Y_n \in \widetilde{\mathcal{M}}^{\mathrm{h}}$ *with* $X_n \leq Y_n \leq Z_n$ *for* $n \in \mathbb{N}$. *Suppose that*

$$X_n \xrightarrow{\tau} X, Y_n \xrightarrow{\tau} Y, Z_n \xrightarrow{\tau} Z \quad \text{and} \quad \tau(X_n) \to \tau(X),\ \tau(Z_n) \to \tau(Z) \quad \text{as } n \to \infty.$$

Then

(i) $Y, Y_n \in L_1(\mathcal{M}, \tau)$ *and* $\tau(Y_n) \to \tau(Y)$ *as* $n \to \infty$;

(ii) *if in addition* $X_n \leq 0 \leq Z_n$ *and* $X_n \leq (Y_n)^p \leq Z_n$, *where* $0 < p < \infty$ *is such that the function* $\mathbb{R} \ni \lambda \mapsto \lambda^p \in \mathbb{R}$ *is defined, then* $Y_n, Y \in L_p(\mathcal{M}, \tau)$ *and* $\|Y_n - Y\|_p \to 0$ *as* $n \to \infty$.

PROOF. (i) Since the cone $\widetilde{\mathcal{M}}^+$ is t_τ-closed, $X \leq Y \leq Z$. Consequently, $-|X| \leq X \leq Y \leq Z \leq |Z|$ and $-(|X| + |Z|) \leq Y \leq |X| + |Z|$. There exists [25] a unitary operator $V \in \mathcal{M}^h$ with

$$|Y| \leq \frac{|X| + |Z| + V(|X| + |Z|)V}{2}.$$

Thus, $Y \in L_1(\mathcal{M}, \tau)$. (This follows from a well-known fact, see [26, Corollary 2] for instance, with $f(t) = \max\{0, t\}$: if $A, B \in \widetilde{\mathcal{M}}^{\mathrm{h}}$ and $A \leq B$ then $\tau(A^+) \leq \tau(B^+)$. Here $A = A^+ - A^-$ is the Jordan decomposition with $A^+, A^- \in \widetilde{\mathcal{M}}^+$ and $A^+A^- = 0$.) Similarly, $Y_n \in L_1(\mathcal{M}, \tau)$ for all $n \in \mathbb{N}$. Use the noncommutative Fatou lemma [20, Theorem 3.5(i)]:

If $T, T_n \in \widetilde{\mathcal{M}}^+$ *and* $T_n \xrightarrow{\tau} T$ *then* $\tau(T) \leq \liminf_{n\to\infty} \tau(T_n)$. We have $Y_n - X_n \xrightarrow{\tau} Y - X$ (as $n \to \infty$) and

$$\tau(Y - X) \leq \liminf_{n\to\infty} \tau(Y_n - X_n) = \liminf_{n\to\infty} \tau(Y_n) - \tau(X);$$

whence, $\tau(Y) \leq \liminf_{n\to\infty} \tau(Y_n)$. Since $Z_n - Y_n \xrightarrow{\tau} Z - Y$ (as $n \to \infty$) and

$$\tau(Z - Y) \leq \liminf_{n\to\infty} \tau(Z_n - Y_n) = \tau(Z) - \limsup_{n\to\infty} \tau(Y_n),$$

we obtain $\tau(Y) \geq \limsup_{n\to\infty} \tau(Y_n)$. Consequently, $\tau(Y) = \lim_{n\to\infty} \tau(Y_n)$.

(ii) Recall [20, Theorem 3.7] that if $A_n, A \in L_1(\mathcal{M}, \tau)$ ($n \in \mathbb{N}$) then

$$\|A_n - A\|_1 \to 0 \iff A_n \xrightarrow{\tau} A \quad \text{and } \|A_n\|_1 \to \|A\|_1$$

as $n \to \infty$. We have

$$\|X\|_1 = -\tau(X), \quad \|Z\|_1 = \tau(Z), \quad \|X_n\|_1 = -\tau(X_n), \quad \|Z_n\|_1 = \tau(Z_n), \quad n \in \mathbb{N}.$$

Thus, $X_n \to X$ and $Z_n \to Z$ as $n \to \infty$ in $L_1(\mathcal{M}, \tau)$. Since $-(Z_n - X_n) \leq (Y_n)^p \leq Z_n - X_n$ for $n \in \mathbb{N}$ and $Z_n - X_n \to Z - X$ as $n \to \infty$ in $L_1(\mathcal{M}, \tau)$, condition (ii) of Theorem 3.1 of [27] is satisfied with $b_n = Z_n - X_n$ and $a_n = Y_n$ for $n \in \mathbb{N}$ and $\sigma = p$. Consequently, $Y_n, Y \in L_p(\mathcal{M}, \tau)$ and $\|Y_n - Y\|_p \to 0$ as $n \to \infty$. □

The authors are grateful to O. E. Tikhonov, A. N. Sherstnev, S. V. Astashkin, and F. A. Sukochev for useful discussion and to the referee for valuable advice.

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