ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2015, Vol. 59, No. 5, pp. 58–63. © Allerton Press, Inc., 2015.
Original Russian Text © A.M. Bikchentaev, 2015, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2015, No. 5, pp. 69–74.

Ideal F-Norms on C^*-Algebras

A. M. Bikchentaev[1*]

[1]*Kazan (Volga Region) Federal University, ul. Kremlyovskaya 18, Kazan, 420008 Russia*

Received October 13, 2014

Abstract—We show that every measure of non-compactness on a W^*-algebra is an ideal F-pseudonorm. We establish a criterion of the right Fredholm property of an element with respect to a W^*-algebra. We prove that the element $-I$ realizes the maximum distance from a positive element to a subset of all isometries of a unital C^*-algebra, here I is the unit of the C^*-algebra. We also consider differences of two finite products of elements from the unit ball of a C^*-algebra and obtain an estimate of their ideal F-pseudonorms. We conclude the paper with a convergence criterion in complete ideal F-norm for two series of elements from a W^*-algebra.

DOI: 10.3103/S1066369X15050084

Keywords: *C^*-algebra, W^*-algebra, trace, Hilbert space, linear operator, Fredholm operator, isometry, unitary operator, compact operator, ideal, ideal F-norm, measure of non-compactness.*

Introduction. We study ideal F-norms on C^*-algebras. We show that every measure of non-compactness on a W^*-algebra is an ideal F-pseudonorm. We establish a criterion of the right Fredholm property of an element with respect to a W^*-algebra. We prove that the minimum distance with respect to an ideal seminorm from an arbitrary element to the Hermitian (respectively, skew-Hermitian) part of a C^*-algebra is realized on the Hermitian (respectively, skew-Hermitian) part of this element. We show that the maximum of the distance with respect to an ideal F-pseudonorm from a positive element to the subset of all isometries of a unital C^*-algebra is realized on the element $-I$. We obtain an estimate of an ideal F-pseudonorm of the difference of two finite products of elements of a unit ball of a C^*-algebra. We establish a convergence criterion with respect to a complete ideal F-norm for two series consisting of elements of a W^*-algebra.

1. Definitions and notations. A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. A W^*-algebra is a C^*-algebra $\mathcal{A}$, that has a predual Banach space $\mathcal{A}_*$: $\mathcal{A} \simeq (\mathcal{A}_*)^*$. For a C^*-algebra $\mathcal{A}$, let $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ denote its subsets of Hermitian elements and positive elements, respectively. Let $\mathcal{A}^1 = \{A \in \mathcal{A} : \|A\| \le 1\}$. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$, $\Re A = (A + A^*)/2$ and $\Im A = (A - A^*)/(2i)$ lie in $\mathcal{A}^{\mathrm{sa}}$. For a unital $\mathcal{A}$, let $\mathcal{A}^{\mathrm{u}}$ and $\mathcal{A}^{\mathrm{iso}}$ denote its subsets of unitary elements ($A^*A = AA^* = I$) and isometries ($A^*A = I$), respectively.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be a W^*-algebra of all linear bounded operators in $\mathcal{H}$. Any C^*-algebra can be realized as a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (I. M. Gel'fand–M. A. Naimark; see [1], theorem 3.4.1).

Let $\mathcal{A}$ be a W^*-algebra. For projectors $P, Q \in \mathcal{A}$, let us write $P \sim Q$ if $P = U^*U$ and $Q = UU^*$ with some $U \in \mathcal{A}$. A projector $P \in \mathcal{A}$ is called *finite*, if $P \sim Q \le P$ implies $P = Q$; $\mathcal{A}$ is called *finite*, if the projector I is finite. Let $\mathcal{F}$ denote an ideal generated by finite, with respect to $\mathcal{A}$, projectors. A uniform closure of $\mathcal{F}$ forms an ideal $\mathcal{K}$ of compact (with respect to $\mathcal{A}$) elements. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{K}$ be a canonical mapping. An element $A \in \mathcal{A}$ is called right Fredholm with respect to $\mathcal{A}$, if $\pi(A)$ is right invertible in $\mathcal{A}/\mathcal{K}$. Let us denote the set of all such elements as $\Phi^-(\mathcal{A})$.

*E-mail: Airat.Bikchentaev@kpfu.ru.

2. Main results. Let $\mathcal{A}$ be a C^*-algebra.

Lemma 1 ([1], theorem 2.2.5, (2)). *If $A, B \in \mathcal{A}^{\mathrm{sa}}$ and $C \in \mathcal{A}$, then the inequality $A \le B$ implies $CAC^* \le CBC^*$.*

Lemma 2 (ibid., theorem 2.2.6). *If $A, B \in \mathcal{A}^+$, then the inequality $A \le B$ implies $\sqrt{A} \le \sqrt{B}$.*

Lemma 3. *If $A, B \in \mathcal{A}$, then $|BA| \le \|B\|\,|A|$.*

Definition 1. A mapping $\rho : \mathcal{A} \to [0, +\infty]$ is called an ideal F-pseudonorm, if $\rho(0) = 0$ and the following conditions are fulfilled:

(i) $\rho(A) = \rho(A^*) = \rho(|A|)$ for all $A \in \mathcal{A}$,

(ii) $\rho(A) \le \rho(B)$ for all $A, B \in \mathcal{A}^+$ with $A \le B$,

(iii) $\rho(A + B) \le \rho(A) + \rho(B)$ for all $A, B \in \mathcal{A}$.

In addition, the set $J_\rho = \{A \in \mathcal{A} : \rho(A) < +\infty\}$ is a $*$-ideal in $\mathcal{A}$. For example, if $A \in J_\rho$ and $B \in \mathcal{A}$, then by Lemma 3 we have

$$\rho(BA) = \rho(|BA|) \le \rho(\|B\|\,|A|) \le \rho(([\|B\|] + 1)|A|) \le ([\|B\|] + 1)\rho(|A|) < +\infty,$$

where $[a]$ is the integer part of the number a. The following conditions are natural:

(iv) $\rho(\varepsilon A) \to 0$ $(\varepsilon \to 0+)$ for all $A \in J_\rho \bigcap \mathcal{A}^+$,

(v) $\rho(A^*A) = \rho(AA^*)$ for all $A \in \mathcal{A}$.

A mapping $\rho : \mathcal{A} \to [0, +\infty]$ is called an ideal F-norm, if $\rho(A) = 0 \Longleftrightarrow A = 0$ and conditions (i)–(iv) are fulfilled. If $\mathcal{A}$ is unital and $\rho : \mathcal{A} \to \mathbb{R}^+$ satisfies condition (ii), then (iv) is equivalent to the condition

(iv)$'$ $\rho(\varepsilon I) \to 0$ $(\varepsilon \to 0+)$,

since $0 \le \varepsilon A \le \varepsilon\|A\|I$ for all $\varepsilon > 0$ and $A \in \mathcal{A}^+$, and we have $\rho(0) = 0$.

For W^*-algebras $\mathcal{A}$ mappings $\rho : \mathcal{A} \to [0, +\infty]$ with properties (i)–(iii) are studied in [2–4]. For a broad class of mappings $\rho : \mathcal{A}^+ \to [0, +\infty]$ with properties (ii), (v) and

(iii)$'$ $\rho(A + B) \le \rho(A) + \rho(B)$ for all $A, B \in \mathcal{A}^+$

representations through positive elements of $\mathcal{A}_*$ are obtained: in [5] for Abelian $\mathcal{A}$ and in [6] for atomic $\mathcal{A}$.

Lemma 4. *Let $\mathcal{A}$ be a unital C^*-algebra and $\rho : \mathcal{A} \to [0, +\infty]$ satisfy condition* (i). *Then $\rho(A) = \rho(UAV^*)$ for all $A \in \mathcal{A}$ and $U, V \in \mathcal{A}^{\mathrm{iso}}$. If $\mathcal{A}$ is a W^*-algebra and ρ additionally satisfies condition* (ii)*, then ρ satisfies* (v).

Proof. We have $|UX| = |X|$ for all $X \in \mathcal{A}$ and $U \in \mathcal{A}^{\mathrm{iso}}$. Let $B = AV^*$, then $\rho(UAV^*) = \rho(UB) = \rho(|UB|) = \rho(B) = \rho(|B^*|) = \rho(|VA^*|) = \rho(|A^*|) = \rho(A)$.

Let $\mathcal{A}$ be a W^*-algebra and ρ satisfy (i) and (ii), $A \in \mathcal{A}$ and $A^* = U|A^*|$ be a polar decomposition. Then $U \in \mathcal{A}^1$ and $|A^*| \in \mathcal{A}^+$, $|A| = U|A^*|U^*$ and $A^*A = UAA^*U^*$. Let $B = AA^*U^*$, then $|UB| \le |B|$ by Lemma 3. We have

$$\rho(A^*A) = \rho(UB) = \rho(|UB|) \le \rho(|B|) = \rho(B) = \rho(AA^*U^*) = \rho(|(AA^*U^*)^*|) = \rho(|UAA^*|) \le \rho(AA^*).$$

Changing A by A^*, in view of the equality $(A^*)^* = A$, we get $\rho(AA^*) \le \rho(A^*A)$ for all $A \in \mathcal{A}$. □

Definition 2 ([7], definition 2.1). Let $\mathcal{A}$ be a W^*-algebra. A mapping $\delta : \mathcal{A} \to \mathbb{R}^+$ is called a measure of non-compactness, if the following conditions are fulfilled:

(a) δ is a seminorm on $\mathcal{A}$,

(b) $\delta(A) = 0 \Longleftrightarrow A \in \mathcal{K}$,

(c) $\delta(A) \le \|A\|$ for all $A \in \mathcal{A}$,

(d) $\delta(AB) \le \delta(A)\delta(B)$ for all $A, B \in \mathcal{A}$.

For example, $\alpha(A) = \inf\{\|A - K\| \mid K \in \mathcal{K}\}$ is a measure of non-compactness on $\mathcal{A}$. It is well-known that the Calkin algebra $\mathcal{A}/\mathcal{K}$ with respect to a norm induced by α is a C^*-algebra. Since $\delta(A) = \delta(A + K)$ for all $K \in \mathcal{K}$ and measures of non-compactness δ, (c) implies $\delta(A) \le \alpha(A)$ for all $A \in \mathcal{A}$.

Proposition 1. *Every measure of non-compactness δ on a W^*-algebra $\mathcal{A}$ satisfies conditions* (i)–(v).

To verify (i), we note that for $A \in \mathcal{A}$ the equality $\delta(A) = \delta(|A|)$ is given in [7] (P. 366, remark 4). If $A = U|A|$ is a polar decomposition, then $U \in \mathcal{A}^1$ and $\delta(A^*) = \delta(|A|U^*) \le \delta(|A|)\delta(U^*) \le \|U^*\|\delta(A) \le \delta(A)$. Changing the places of A and A^*, we get $\delta(A) \le \delta(A^*)$.

To verify (ii), we pick $A, B \in \mathcal{A}^+$ with $A \le B$. Then there exists an element $C \in \mathcal{A}^1$ such that $A = CBC^*$ ([8], Chap. 1, Section 1, lemma 2). By (d) and (c) we have

$$\delta(A) = \delta(CBC^*) \le \|C\|\,\|C^*\|\delta(B) \le \delta(B).$$

Properties (iii) and (iv) follow from (a); now (v) follows from Lemma 4. □

From theorem 2.4 in [7] and Proposition 1 we get

Corollary 1. Let δ be a measure of non-compactness on a W^*-algebra $\mathcal{A}$ and $A \in \mathcal{A}$. Any element $A \in \Phi^-(\mathcal{A})$ if and only if there exists a constant $c > 0$ such that $\delta(BA) \ge c\delta(B)$ for all $B \in \mathcal{A}$.

Let us note that in [7] (P. 367) the statement was given with an "additional" condition of $\delta(T) = \delta(T^*)$, $T \in \mathcal{A}$.

Lemma 5. *Let $\mathcal{A}$ be a C^*-algebra and $\rho : \mathcal{A} \to [0, +\infty]$ satisfy conditions* (ii) *and* (v). *Then $\rho(\sqrt{A_1}A_2\sqrt{A_1}) \le \rho(\sqrt{A_2}B_1\sqrt{A_2}) \le \rho(\sqrt{B_1}B_2\sqrt{B_1})$ for all $A_k, B_k \in \mathcal{A}$ with $A_k \le B_k$, $k = 1, 2$.*

Lemma 1 yields $\sqrt{A_2}A_1\sqrt{A_2} \le \sqrt{A_2}B_1\sqrt{A_2}$ and $\sqrt{B_1}A_2\sqrt{B_1} \le \sqrt{B_1}B_2\sqrt{B_1}$, hence

$$\rho(\sqrt{A_1}A_2\sqrt{A_1}) = \rho(\sqrt{A_2}A_1\sqrt{A_2}) \le \rho(\sqrt{A_2}B_1\sqrt{A_2}) = \rho(\sqrt{B_1}A_2\sqrt{B_1}) \le \rho(\sqrt{B_1}B_2\sqrt{B_1}).$$

Proposition 2. *Let $\mathcal{A}$ be a unital C^*-algebra and $\rho : \mathcal{A} \to [0, +\infty]$ satisfy condition* (ii). *Then $\rho(A + B) \le \rho(\sqrt{I + B}(I + A)\sqrt{I + B})$ for all $A \in \mathcal{A}^+ \bigcap \mathcal{A}^1$ and $B \in \mathcal{A}^+$. If, in addition, ρ satisfies condition* (v)*, then $\rho(\sqrt{I + B}(I + A)\sqrt{I + B}) \le \rho(e^{B/2}e^Ae^{B/2})$ for all $A, B \in \mathcal{A}^+$.*

Proof. Since $0 \le A \le I$, by Lemma 1 we have

$$A + B \le I + B + \sqrt{I + B}A\sqrt{I + B} = \sqrt{I + B}(I + A)\sqrt{I + B}.$$

Since $I + X \le e^X$ for all $X \in \mathcal{A}^+$, we can apply Lemma 5. □

Proposition 3. *Let $\mathcal{A}$ be a C^*-algebra, $A \in \mathcal{A}$, a mapping $\rho : \mathcal{A} \to [0, +\infty]$ satisfy condition* (iii) *and $\rho(X) = \rho(-X) = \rho(X^*) = 2\rho(X/2)$ for all $X \in \mathcal{A}$. Then $\rho(A - \Re A) \le \rho(A - B)$ and $\rho(A - i\Im A) \le \rho(A - iB)$ for all $B \in \mathcal{A}^{\mathrm{sa}}$.*

Thus, $\inf_{B \in \mathcal{A}^{\mathrm{sa}}} \rho(A - B) = \rho(A - \Re A)$ and $\inf_{B \in \mathcal{A}^{\mathrm{sa}}} \rho(A - iB) = \rho(A - i\Im A)$ for all $A \in \mathcal{A}$. The statement follows from the equalities

$$A - \Re A = \frac{A - B}{2} - \frac{A^* - B}{2} = \frac{A - B}{2} - \frac{(A - B)^*}{2},$$
$$A - i\Im A = \frac{A - iB}{2} + \frac{A^* + iB}{2} = \frac{A - iB}{2} + \frac{(A - iB)^*}{2}.$$ □

Theorem 1. *Let $\mathcal{A}$ be a unital C^*-algebra and $\rho : \mathcal{A}^+ \to \mathbb{R}^+$ satisfy conditions* (ii), (iii)′, (iv)′ *and* (v). *Then $\rho(|A - U|) \le \rho(A + I)$ for all $A \in \mathcal{A}^+$ and $U \in \mathcal{A}^{\mathrm{iso}}$.*

Proof. By theorem 4.2 in [9] we get

$$\forall\varepsilon > 0\, \exists V, W \in \mathcal{A}^{\mathrm{u}}\ \ (|A - U| \le VAV^* + W|U|W^* + \varepsilon I = V(A + I)V^* + \varepsilon I).$$

By the properties of ρ and Lemma 4, we get $\rho(|A - U|) \le \rho(V(A + I)V^* + \varepsilon I) \le \rho(A + I) + \rho(\varepsilon I)$. We complete the proof by passing to the limit as $\varepsilon \to 0+$. □

Thus, $\sup\limits_{U\in\mathcal{A}^{\mathrm{iso}}} \rho(|A - U|) = \rho(A - (-I))$ for all $A \in \mathcal{A}^+$. In other words, the maximal "ρ-distance" from an element $A \in \mathcal{A}^+$ to the set $\mathcal{A}^{\mathrm{iso}}$ is realized on the element $U_0 = -I$. Since $U_0 \in \mathcal{A}^{\mathrm{u}}$, we have $\sup\limits_{U\in\mathcal{A}^{\mathrm{u}}} \rho(|A - U|) = \rho(A - (-I))$.

Let J be a $*$-ideal in a unital C^*-algebra $\mathcal{A}$ and $A \in \mathcal{A}^+$. If $U - A \in J$ for some $U \in \mathcal{A}^{\mathrm{iso}}$, then $I - A \in J$. Indeed, we have $U^* - A \in J$ and $I - A^2 = (U^* - A)(U + A) + U^*(U - A) - (U^* - A)U \in J$. Since $I + A$ is invertible, we have $I - A = (I - A^2)(I + A)^{-1} \in J$.

Corollary 2. Let $\mathcal{A}$ be a unital C^*-algebra and $\rho : \mathcal{A} \to \mathbb{R}^+$ satisfy conditions (i)–(iii), (iv)$'$ and (v). If $A \in \mathcal{A}$ has a polar decomposition $A = U|A|$ with $U \in \mathcal{A}^{\mathrm{u}}$, then

$$\sup_{V\in\mathcal{A}^{\mathrm{iso}}} \rho(A - V) = \sup_{V\in\mathcal{A}^{\mathrm{u}}} \rho(A - V) = \rho(A + U).$$

Proof. For $V \in \mathcal{A}^{\mathrm{iso}}$ we have $U^*V \in \mathcal{A}^{\mathrm{iso}}$. By Lemma 4 and Theorem 1, we get

$$\begin{aligned}\rho(A - V) = \rho(U|A| - V) = \rho(U(|A| - U^*V)) = \rho(|A| - U^*V)&\\ \le \rho(|A| + I) = \rho(U|A| + U) = \rho(A + U).&\end{aligned}$$ □

If $\mathcal{A}$ is a finite W^*-algebra, $A \in \mathcal{A}$ and $A = T|A|$ is a polar decomposition with a partial isometry T, then T can be extended to $U \in \mathcal{A}^{\mathrm{u}}$ with the property $A = U|A|$ (see [3], proof of theorem 2).

Theorem 2. *Let $\mathcal{A}$ be a C^*-algebra and $\rho : \mathcal{A} \to [0, +\infty]$ satisfy conditions* (i)–(iii). *Then*

$$\rho\Big(\prod_{k=1}^{n} A_k - \prod_{k=1}^{n} B_k\Big) \le \sum_{k=1}^{n} \rho(A_k - B_k)\ \ \textit{for all}\ \ A_k, B_k \in \mathcal{A}^1,\ \ k = 1, \dots, n. \tag{1}$$

Proof. By Lemmas 1–3 we get

$$|((A_1 - B_1)A_2)^*| = \sqrt{(A_1 - B_1)A_2A_2^*(A_1 - B_1)^*} \le |(A_1 - B_1)^*|,$$
$$|B_1(A_2 - B_2)| = \sqrt{(A_2 - B_2)^*B_1^*B_1(A_2 - B_2)} \le |A_2 - B_2|.$$

Let us carry out an induction with respect to $n \in \mathbb{N}$. For $n = 2$ we have

$$\rho(A_1A_2 - B_1B_2) = \rho((A_1 - B_1)A_2 + B_1(A_2 - B_2)) \le \rho(A_1 - B_1) + \rho(A_2 - B_2).$$

Induction hypothesis: let (1) be fulfilled for all $n = 1, 2, \dots, m$. Then

$$\rho\Big(\prod_{k=1}^{m+1} A_k - \prod_{k=1}^{m+1} B_k\Big) \le \rho\Big(\prod_{k=1}^{m} A_k - \prod_{k=1}^{m} B_k\Big) + \rho(A_{m+1} - B_{m+1}) \le \sum_{k=1}^{m+1} \rho(A_k - B_k).$$ □

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra, $\rho : \mathcal{A} \to [0, +\infty]$ be an ideal F-norm such that J_ρ is complete with respect to the metric $d_\rho(A, B) = \rho(A - B)$, $X_n, Y_n \in \mathcal{A}^{\mathrm{sa}}$ and $Z_n = X_n + iY_n$, $n \in \mathbb{N}$. If the series $\sum\limits_{n=1}^{\infty} X_n^2$ and $\sum\limits_{n=1}^{\infty} Z_n^2$ are ρ-convergent, then the series $\sum\limits_{n=1}^{\infty} |Z_n|^2$ and $\sum\limits_{n=1}^{\infty} |Z_n^*|^2$ are also ρ-convergent; for a W^*-algebra $\mathcal{A}$ the converse is true as well.*

Proof. If $A \in \mathcal{A}$, then $\rho(\Re A) \leq \rho(A) + \rho(A^*) = 2\rho(A)$. Similarly, $\rho(\Im A) \leq 2\rho(A)$. Hence, $\rho(A) \leq \rho(\Re A) + \rho(\Im A) \leq 4\rho(A)$ and the ρ-convergence of the sequence of elements is equivalent to the ρ-convergence of the Hermitian and the skew-Hermitian parts of these elements. Since the series $\sum_{n=1}^{\infty}(X_n^2 - Y_n^2) = \Re \sum_{n=1}^{\infty}(X_n^2 - Y_n^2 + i(X_nY_n + Y_nX_n)) = \Re \sum_{n=1}^{\infty} Z_n^2$ is ρ-convergent, then the series $\sum_{n=1}^{\infty} Y_n^2$ is ρ-convergent, too. Since

$$|Z_n|^2 + |Z_n^*|^2 = 2X_n^2 + 2Y_n^2, \ \ n \in \mathbb{N}, \tag{2}$$

the series $\sum_{n=1}^{\infty}|Z_n|^2$ and $\sum_{n=1}^{\infty}|Z_n^*|^2$ are ρ-convergent as well.

Let now $\mathcal{A}$ be a W^*-algebra and the series $\sum_{n=1}^{\infty}|Z_n|^2$ and $\sum_{n=1}^{\infty}|Z_n^*|^2$ be ρ-convergent. By (2), the series $\sum_{n=1}^{\infty} X_n^2$ and $\sum_{n=1}^{\infty} Y_n^2$ are also ρ-convergent. Hence,

$$\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N} \ \ \forall k \geq N, \ \ \forall m \in \mathbb{N} \ \ \left(\rho\biggl(\sum_{n=k}^{k+m}(X_n^2 + Y_n^2)\biggr) < \varepsilon\right). \tag{3}$$

Let $\varepsilon > 0$ and k, m be chosen in (3). Assume

$$A_{k,m} = \sum_{n=k}^{k+m}(X_n^2 + Y_n^2), \quad B_{k,m} = \sum_{n=k}^{k+m}(X_nY_n + Y_nX_n).$$

Since $(X_n \pm Y_n)^2 \geq 0$, we get $-(X_n^2 + Y_n^2) \leq X_nY_n + Y_nX_n \leq X_n^2 + Y_n^2$. By conducting a termwise summation of these double inequalities over all $n = k, \ldots, k+m$, we get $-A_{k,m} \leq B_{k,m} \leq A_{k,m}$. By theorem 1 in [4] and by [10] there exists an element $S \in \mathcal{A}^{\mathrm{u}} \bigcap \mathcal{A}^{\mathrm{sa}}$ such that $2|B_{k,m}| \leq A_{k,m} + SA_{k,m}S$. Then $S^2 = I$ and by the definition of ρ, Lemma 4 and (3) we have

$$\begin{aligned}\rho(B_{k,m}) &\leq \rho(2|B_{k,m}|) \leq \rho(A_{k,m} + SA_{k,m}S) \\ &\leq \rho(A_{k,m}) + \rho(\sqrt{A_{k,m}}S^2\sqrt{A_{k,m}}) = 2\rho(A_{k,m}) < 2\varepsilon.\end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty}(X_nY_n + Y_nX_n)$ is ρ-convergent. □

Example. Let τ be a faithful normal semifinite trace on a W^*-algebra $\mathcal{A}$ and a number $p \in (0, +\infty)$. Define the mapping $\rho : \mathcal{A} \to [0, +\infty]$ as

$$\rho(A) = \begin{cases} \tau(|A|^p)^{1/p}, & \text{if } p > 1; \\ \tau(|A|^p), & \text{if } 0 < p \leq 1. \end{cases}$$

Then ρ satisfies conditions (i)–(v). If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$ is a canonical trace, then J_ρ coincides with the Schatten–von Neumann ideal $\mathfrak{S}_p$. The operator $A \in \mathcal{B}(\mathcal{H})$ has a finite order, if $A \in \mathfrak{S}_p$ for some $p > 0$. The lower bound of the values of p, for which this relation holds, is called the order of the operator and is denoted as $q(A)$, i.e., $q(A) = \inf\{p > 0 \mid A \in \mathfrak{S}_p\}$. Thus, $q(A + B) \leq \max\{q(A), q(B)\}$ for all $A, B \in \mathcal{B}(\mathcal{H})$ and q is an ideal F-pseudonorm, q does not satisfy (iv). We get $J_q = \bigcup_{p>0} \mathfrak{S}_p$.

REFERENCES

1. Murphy, G. J. *C^*-Algebras and Operator Theory* (Academic Press, 1990; Faktorial, Moscow, 1997).
2. Bikchentaev, A. M. "The Triangle Inequality for Some Spaces of Measurable Operators," in *Konstr. Teor. Funkts. Funkts. Analiz* (Kazan University Press, Kazan, 1992) No. 8, 23–32.
3. Bikchentaev, A. M. "Minimality of Convergence in Measure Topologies on Finite von Neumann Algebras," Mathematical Notes **75**, No. 3–4, 315–321 (2004).

4. Bikchentaev, A. M. “Block Projection Operators in Normed Solid Spaces of Measurable Operators,” Russian Mathematics (Iz. VUZ) **56**, No. 2, 75–79 (2012).
5. Bikchentaev, A. M. “The Haagerup Problem on Subadditive Weights on W^*-Algebras,” Russian Mathematics (Iz. VUZ) **55**, No. 10, 82–85 (2011).
6. Bikchentaev, A. M. “The Haagerup problem on subadditive weights on W^*-algebras. II,” Russian Mathematics (Iz. VUZ) **57**, No. 12, 66–69 (2013).
7. Ströh, A., Swart, J. “Measures of Noncompactness of Operators in von Neumann Algebras,” Indiana Univ. Math. J. **38**, No. 2, 365–375 (1989).
8. Dixmier, J. *Les Algebres d'Opérateurs dans l'Espace Hilbertien* (*Algebres de von Neumann*), 2nd ed. (Gauthier-Villars, Paris, 1969).
9. Akemann, C. A., Anderson, J., Pedersen, G. K. “Triangle Inequalities in Operator Algebras,” Linear Multilinear Algebra **11**, No. 2, 167–178 (1982).
10. Bikchentaev, A. M. “On Hermitian Operators X and Y Meeting the Condition $-Y \leq X \leq Y$,” Lobachevskii J. Math. **34**, No. 3, 227–233 (2013).

Translated by P. A. Novikov