On operators all of which powers have the same trace

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We introduce the class $K_{\mathcal{A},\phi} = \{A \in \mathcal{A} : \phi(A^k) = \phi(A) \ \ \forall k \in \mathbb{N}\}$ for a linear functional ϕ on an algebra $\mathcal{A}$ and consider the properties of this class. We have $K_{\mathcal{A},\phi} = K_{\mathcal{A},t\phi}$, for all $t \in \mathbb{C}$. Also we prove the "0–1 number lemma": if a set $\{z_k\}_{k=1}^n \subset \mathbb{C}$ is such that

$$z_1 + \ldots + z_n = z_1^2 + \ldots + z_n^2 = \cdots = z_1^{n+1} + \ldots + z_n^{n+1},$$

then $z_k \in \{0,1\}$, for all $k = 1, 2, \ldots, n$. This lemma helps us to show that $\{\phi(A) : A \in K_{\mathcal{A},\phi}\} = \{0, 1, \ldots, n\}$ and $\det(A) \in \{0,1\}$ for $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and $\phi = \mathrm{tr}$, the canonical trace. We have $A = P + Z$ where P is a projection and Z is a nilpotent for any $A \in K_{\mathcal{A},\phi}$.

Assume that a trace class operator A on a Hilbert space admits a constant $C \in \mathbb{C}$ such that $\forall k \in \mathbb{N}$ $\mathrm{tr}(A^k) = C$. Then $C \in \mathbb{N} \cup \{0\}$ and the spectrum $\sigma(A)$ is a subset of $\{0,1\}$. Finally we give the description of all the elements of the class $K_{\mathcal{A},\phi}$ for $\mathbb{M}_2(\mathbb{C})$.

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