Siberian Mathematical Journal, *Vol.* 58, *No.* 2, *pp.* 183–189, 2017
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DIFFERENCES OF IDEMPOTENTS IN C^*-ALGEBRAS

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Abstract: Suppose that P and Q are idempotents on a Hilbert space $\mathscr{H}$, while $Q = Q^*$ and I is the identity operator in $\mathscr{H}$. If $U = P - Q$ is an isometry then $U = U^*$ is unitary and $Q = I - P$. We establish a double inequality for the infimum and the supremum of P and Q in $\mathscr{H}$ and $P - Q$. Applications of this inequality are obtained to the characterization of a trace and ideal F-pseudonorms on a W^*-algebra. Let φ be a trace on the unital C^*-algebra $\mathscr{A}$ and let tripotents P and Q belong to $\mathscr{A}$. If $P - Q$ belongs to the domain of definition of φ then $\varphi(P - Q)$ is a real number. The commutativity of some operators is established.

DOI: 10.1134/S003744661702001X

Keywords: Hilbert space, linear operator, idempotent, tripotent, projection, unitary operator, trace class operator, operator inequality, commutativity, W^*-algebra, C^*-algebra, trace, ideal F-norm

Introduction

Let P and Q be idempotents on a Hilbert space $\mathscr{H}$. Various properties (invertibility, Fredholm property, nuclearity, positivity, etc.) of the difference $P - Q$ were studied in [1–6]. Each tripotent ($A = A^3$) is the difference $P - Q$ of some idempotents P and Q with $PQ = QP = 0$ [7, Proposition 1]. Therefore, tripotents inherit some properties of idempotents [8].

In this article, we obtain some new results on $P-Q$. We prove that the isometry of $U = P-Q$, where $Q^* = Q$, implies the unitarity of U and the equality $Q = I-P$ (Theorem 1). We give an example showing the substantiality of the condition $Q^* = Q$. If $P^* = P$ then $P \wedge Q^\perp + P^\perp \wedge Q \le |P-Q| \le P \vee Q - P \wedge Q$ with equality in the second inequality if and only if $PQ = QP$ (Theorem 2 and Proposition 1). Using this operator inequality, we establish a new inequality that characterizes traces on a W^*-algebra $\mathscr{A}$ (Corollary 4). Applications are obtained to ideal F-pseudonorms on $\mathscr{A}$ (Corollary 7).

Let φ be a trace on a unital C^*-algebra $\mathscr{A}$, let $\mathfrak{M}_\varphi$ be the domain of definition of φ, and let P and Q belong to $\mathscr{A}$. If $P - Q \in \mathfrak{M}_\varphi$ then $\varphi(P - Q) \in \mathbb{R}$ (Theorem 3). Theorem 3 is a C^*-analog of the following familiar assertion [6]: *If P and Q are idempotents in $\mathscr{H}$ and $P - Q$ belongs to the ideal $\mathfrak{S}_1$ of trace class operators then the canonical trace* $\operatorname{tr}(P - Q)$ *belongs to* $\mathbb{Z}$. Let $\mathscr{A}$ be a C^*-algebra and let $(\mathscr{E}, \prec)$ be a partially ordered set. We establish a monotonicity criterion for a mapping from $\mathscr{A}^+$ into $\mathscr{E}$ (Proposition 2).

1. Definitions and Notations

By a C^*-*algebra* we mean a complex Banach $*$-algebra $\mathscr{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathscr{A}$. Given a C^*-algebra $\mathscr{A}$, denote by $\mathscr{A}^{\mathrm{id}}$, $\mathscr{A}^{\mathrm{sa}}$, and $\mathscr{A}^+$ the sets of its idempotents, Hermitian elements, and positive elements respectively. If $A \in \mathscr{A}$ then $|A| = \sqrt{A^*A} \in \mathscr{A}^+$. If $A \in \mathscr{A}^{\mathrm{sa}}$ then $A_+ = (|A| + A)/2$ and $A_- = (|A| - A)/2$ lie in $\mathscr{A}^+$ and $A = A_+ - A_-$, $A_+A_- = 0$. A W^*-*algebra* is a C^*-algebra $\mathscr{A}$ having a predual Banach space $\mathscr{A}_* : \mathscr{A} \simeq (\mathscr{A}_*)^*$. For a W^*-algebra $\mathscr{A}$, denote by $\mathscr{A}^{\mathrm{u}}$ and $\mathscr{A}^{\mathrm{pr}}$ its subsets of unitary elements and the projection lattice respectively. If I is the unity of $\mathscr{A}$ and $P \in \mathscr{A}^{\mathrm{id}}$ then $P^\perp = I - P \in \mathscr{A}^{\mathrm{id}}$.

The author was supported by the Russian Foundation for Basic Research and the Government of the Republic of Tatarstan (Grant 15–41–02433).

Kazan. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 58, No. 2, pp. 243–250, March–April, 2017; DOI: 10.17377/smzh.2017.58.201. Original article submitted March 21, 2016.