Two Classes of τ -Measurable Operators Affiliated with a von Neumann Algebra

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Abstract—Let \mathcal{M} be a von Neumann algebra of operators on a Hilbert space \mathcal{H} , τ be a faithful normal semifinite trace on \mathcal{M} . We define two (closed in the topology of convergence in measure τ) classes \mathcal{P}_1 and \mathcal{P}_2 of τ -measurable operators and investigate their properties. The class \mathcal{P}_2 contains \mathcal{P}_1 . If a τ -measurable operator T is hyponormal, then T lies in \mathcal{P}_1 ; if an operator T lies in \mathcal{P}_k , then UTU^* belongs to \mathcal{P}_k for all isometries U from \mathcal{M} and k = 1, 2; if an operator T from \mathcal{P}_1 admits the bounded inverse T^{-1} , then T^{-1} lies in \mathcal{P}_1 . We establish some new inequalities for rearrangements of operators from \mathcal{P}_1 . If a τ -measurable operator T is hyponormal and T^n is τ -compact for some natural number n, then T is both normal and τ -compact. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{tr}$, then the class \mathcal{P}_1 coincides with the set of all paranormal operators on \mathcal{H} .

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Introduction. Let \mathcal{M} be a von Neumann operator algebra on a Hilbert space \mathcal{H} , τ be a faithful normal semifinite trace on \mathcal{M} , $\widetilde{\mathcal{M}}$ be a *-algebra of all τ -measurable operators, a number 0 $and <math>L_p(\mathcal{M}, \tau)$ be the space of integrable (with respect to τ) with degree p operators. In this paper we introduce two (closed in the topology of convergence in measure τ) classes \mathcal{P}_1 and \mathcal{P}_2 with $\mathcal{P}_1 \subset \mathcal{P}_2$ elements of $\widetilde{\mathcal{M}}$ and investigate their properties. We show that if an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal, then $T \in \mathcal{P}_1$; if an operator $T \in \mathcal{P}_1$, then $UTU^* \in \mathcal{P}_1$ for all isometries $U \in \mathcal{M}$; if an operator $T \in \mathcal{P}_1$ has the inverse $T^{-1} \in \mathcal{M}$, then $T^{-1} \in \mathcal{P}_1$. For $T \in \widetilde{\mathcal{M}}$ we have $T \in \mathcal{P}_2 \Leftrightarrow T^* \in \mathcal{P}_2$. If $T \in \widetilde{\mathcal{M}}$ and the operators $U, V \in \mathcal{M}$ are isometries, then the rearrangement relation $\mu_t(UTV^*) = \mu_t(T)$ holds for all t > 0; if $T \in \mathcal{P}_2$, then $UTU^* \in \mathcal{P}_2$. Let $T \in \widetilde{\mathcal{M}}$ and a unitary operator $S \in \mathcal{M}^{\mathrm{sa}}$ be so that ST = TS. Then $T \in \mathcal{P}_k \Leftrightarrow ST \in \mathcal{P}_k, k = 1, 2$.

If an operator $T \in \mathcal{P}_1$, then $T^2 \in \mathcal{P}_1$ and $\mu_t(T^2) \ge \mu_t(T)^2$ for all t > 0. If an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$, then $T^n \in \mathcal{P}_1$ for all $n \in \mathbb{N}$. The set $\mathcal{P}_1 \cap \mathcal{M}$ is $\|\cdot\|$ -closed in \mathcal{M} . Consider an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ and $n \in \mathbb{N}$. Then $\mu_t(T^n) \ge \mu_t(T)^n$ for all t > 0 and we have the equivalences $T \in \mathcal{F}(\mathcal{M}) \Leftrightarrow T^n \in \mathcal{F}(\mathcal{M})$; $T \in \widetilde{\mathcal{M}}_0 \Leftrightarrow T^n \in \widetilde{\mathcal{M}}_0$; $T \in L_{pn}(\mathcal{M}, \tau) \Leftrightarrow T^n \in L_p(\mathcal{M}, \tau), 0 . Every operator <math>T \in \mathcal{P}_1 \cap \mathcal{M}$ is normaloid. If an operator $(0 \neq)T \in \mathcal{M}$ is quasi-nilpotent, then $T \notin \mathcal{P}_1$. The class \mathcal{P}_2 does not contain either non-selfadjoint symmetries $(T^2 = I)$ or non-selfadjoint idempotents $(T^2 = T)$. If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal and $T^n \in \widetilde{\mathcal{M}}_0$ for some $n \in \mathbb{N}$, then T belongs to $\widetilde{\mathcal{M}}_0$ and is normal.

Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{M}$. We have that $T \in \mathcal{P}_1 \Leftrightarrow T$ is paranormal. The class \mathcal{P}_1 is sequentially closed in the strong operator topology and contains a non-hyponormal operator. If \mathcal{H} is separable and infinite dimensional, then $\mathcal{P}_1 \neq \mathcal{P}_2$.

1. Definition and notation. Let \mathcal{M} be a von Neumann operator algebra on a Hilbert space $\mathcal{H}, \mathcal{M}^{\text{pr}}$ be the lattice of projections on $\mathcal{M}, \mathcal{M}^+$ be the positive element cone in \mathcal{M} . Let I be the identity of \mathcal{M} and $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\text{pr}}$.

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