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On Idempotent τ-Measurable Operators Affiliated to a von Neumann Algebra

A. M. Bikchentaev[*]

Kazan (Volga Region) Federal University, Kazan, Russia

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Abstract—Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$, let p, $0 < p < \infty$, be a number, and let $L_p(\mathscr{M}, \tau)$ be the space of operators whose pth power is integrable (with respect to τ). Let P and Q be τ-measurable idempotents, and let $A \equiv P - Q$. In this case, 1) if $A \geq 0$, then A is a projection and $QA = AQ = 0$; 2) if P is quasinormal, then P is a projection; 3) if $Q \in \mathscr{M}$ and $A \in L_p(\mathscr{M}, \tau)$, then $A^2 \in L_p(\mathscr{M}, \tau)$. Let n be a positive integer, $n > 2$, and $A = A^n \in \mathscr{M}$. In this case, 1) if $A \neq 0$, then the values of the nonincreasing rearrangement $\mu_t(A)$ belong to the set $\{0\} \cup [\|A^{n-2}\|^{-1}, \|A\|]$ for all $t > 0$; 2) either $\mu_t(A) \geq 1$ for all $t > 0$ or there is a $t_0 > 0$ such that $\mu_t(A) = 0$ for all $t > t_0$. For every τ-measurable idempotent Q, there is a unique rank projection $P \in \mathscr{M}$ with $QP = P$, $PQ = Q$, and $P\mathscr{M} = Q\mathscr{M}$. There is a unique decomposition $Q = P + Z$, where $Z^2 = 0$, $ZP = 0$, and $PZ = Z$. Here, if $Q \in L_p(\mathscr{M}, \tau)$, then P is integrable, and $\tau(Q) = \tau(P)$ for $p = 1$. If $A \in L_1(\mathscr{M}, \tau)$ and if $A = A^3$ and $A - A^2 \in \mathscr{M}$, then $\tau(A) \in \mathbb{R}$.

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INTRODUCTION

Let $\mathscr{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathscr{H}$, let τ be a faithful normal semifinite trace on $\mathscr{M}$, let p be a number, $0 < p < \infty$, and let $L_p(\mathscr{M}, \tau)$ be the space of p-integrable operators (with respect to τ). In the paper, we obtain the following results concerning algebraic and order properties of the trace τ and of elements of the $*$-algebra $\widetilde{\mathscr{M}}$ of all τ-measurable operators.

Let $P, Q \in \widetilde{\mathscr{M}}$ be idempotents. If $A \equiv P - Q \geq 0$, then A is a projection and $QA = AQ = 0$ (Theorem 2.5); if P is quasinormal, then P is a projection (Theorem 2.10). If $Q \in \mathscr{M}$ is an idempotent and $A \equiv P - Q \in L_p(\mathscr{M}, \tau)$, then $A^2 \in L_p(\mathscr{M}, \tau)$ (Theorem 2.30). If $A \in L_1(\mathscr{M}, \tau)$ and if $A = A^3$ and $A - A^2 \in \mathscr{M}$, then $\tau(A) \in \mathbb{R}$ (Corollary 2.31).

Let n be a positive integer, $n > 2$. If $A \in \mathscr{M}$ and $0 \neq A = A^n$, then the values of the nonincreasing rearrangement $\mu_t(A)$ belong to the set $\{0\} \cup [\|A^{n-2}\|^{-1}, \|A\|]$ for all $t > 0$ (Theorem 2.13). Let $\widetilde{\mathscr{M}_0}$ be the ideal of τ-compact operators in $\widetilde{\mathscr{M}}$, and let $\mathscr{F}(\mathscr{M})$ be the ideal of elementary operators in $\mathscr{M}$. If $A = A^n \in \mathscr{M} \cap \widetilde{\mathscr{M}_0}$, then $A \in \mathscr{F}(\mathscr{M})$ (Corollary 2.14).

Let $A = A^n \in \mathscr{M}$. Then either $\mu_t(A) \geq 1$ for all $t > 0$ (for $A \notin \widetilde{\mathscr{M}_0}$) or there is a $t_0 > 0$ such that $\mu_t(A) = 0$ for all $t > t_0$ (for $A \in \widetilde{\mathscr{M}_0}$) (Corollary 2.15).

Let $A \in \widetilde{\mathscr{M}}$. Then $A \in \widetilde{\mathscr{M}_0}$ if and only if the real part of A^2 and the imaginary part of A belong to $\widetilde{\mathscr{M}_0}$ (Theorem 2.18). A similar assertion holds also for the ideal $\mathscr{F}(\mathscr{M})$ (Theorem 2.19).

For every idempotent $Q \in \widetilde{\mathscr{M}}$, there is a unique projection $P \in \mathscr{M}$ such that $QP = P$, $PQ = Q$, and $P\mathscr{M} = Q\mathscr{M}$ (Theorem 2.21; we call P the *rank projection*). There is a unique decomposition

[*]E-mail: Airat.Bikchentaev@kpfu.ru