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Integrable Products of Measurable Operators

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Abstract—Let τ be a faithful normal semifinite trace on von Neumann algebra $\mathcal{M}$, $0 < p < +\infty$ and $L_p(\mathcal{M}, \tau)$ be the space of all integrable (with respect to τ) with degree p operators, assume also that $\widetilde{\mathcal{M}}$ is the *-algebra of all τ-measurable operators. We give the sufficient conditions for integrability of operator product $A, B \in \widetilde{\mathcal{M}}$. We prove that $AB \in L_p(\mathcal{M}, \tau) \Leftrightarrow |A|B \in L_p(\mathcal{M}, \tau) \Leftrightarrow |A||B^*| \in L_p(\mathcal{M}, \tau)$; moreover, $||AB||_p = |||A|B||_p = |||A||B^*|||_p$. If A is hyponormal, B is cohyponormal and $AB \in L_p(\mathcal{M}, \tau)$ then $BA \in L_p(\mathcal{M}, \tau)$ and $||BA||_p \le ||AB||_p$; for $p = 1$ we have $\tau(AB) = \tau(BA)$. A nonzero hyponormal (or cohyponormal) operator $A \in \widetilde{\mathcal{M}}$ cannot be nilpotent. If $A \in \widetilde{\mathcal{M}}$ is quasinormal then the arrangement $\mu_t(A^n) = \mu_t(A)^n$ for all $n \in \mathbb{N}$ and $t > 0$. If A is a τ-compact operator and $B \in \widetilde{\mathcal{M}}$ is such that $|A| \log^+ |A|, e^{p|B|} \in L_1(\mathcal{M}, \tau)$ then $AB, BA \in L_1(\mathcal{M}, \tau)$.

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INTRODUCTION

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{M}$. Assume also that $\widetilde{\mathcal{M}}$ is the *-algebra of all τ-measurable operators, consider a number $0 < p < +\infty$ and let $L_p(\mathcal{M}, \tau)$ be the space of all integrable (with respect to τ) with degree p operators.

Integrability of τ-measurable operator product is one of the key problems in the noncommutative integration theory (see, for example, [2, 3, 7, 8, 11, 16–19]). The Golden–Tompson and the Peierls–Bogoliubov inequalities also comprise τ-measurable operator products integration, moreover, these inequalities characterize tracial functionals on C^*-algebras [4, 5]. It is well known in Probability Theory that the product $\xi\eta$ of independent integrable random variables ξ and η is also integrable, moreover, $\int_\Omega \xi\eta d\mathbb{P} = \int_\Omega \xi d\mathbb{P} \int_\Omega \eta d\mathbb{P}$.

In this paper we give the sufficient conditions for integrability of the operator product $A, B \in \widetilde{\mathcal{M}}$. We prove that $AB \in L_p(\mathcal{M}, \tau) \Leftrightarrow |A|B \in L_p(\mathcal{M}, \tau) \Leftrightarrow |A||B^*| \in L_p(\mathcal{M}, \tau)$; moreover, $||AB||_p = |||A|B||_p = |||A||B^*|||_p$. We have $\tau(|A|B) = \tau(B|A|B) = \tau(B|A|)$ and $\tau(B|A|B^\perp) = \tau(B^\perp|A|B) = 0$ for $B \in \mathcal{M}^{\mathrm{pr}}$ and $p = 1$ (Theorem 2.1). If A is hyponormal, B is cohyponormal, and $AB \in L_p(\mathcal{M}, \tau)$ then $BA \in L_p(\mathcal{M}, \tau)$ and $||BA||_p \le ||AB||_p$; we have $\tau(AB) = \tau(BA)$ for $p = 1$ (Theorem 2.3). A nonzero hyponormal (or cohyponormal) operator $A \in \widetilde{\mathcal{M}}$ cannot be nilpotent (Theorem 2.4). If $A \in \widetilde{\mathcal{M}}$ is quasinormal then $\mu_t(A^n) = \mu_t(A)^n$ for all $n \in \mathbb{N}$ and $t > 0$ (Theorem 2.6). If $A \in \widetilde{\mathcal{M}}$ is a quasinormal operator and $A^2 = A$ then $A \in \mathcal{M}^{\mathrm{pr}}$ (Corollary 2.8). If A is a τ-compact operator and $B \in \widetilde{\mathcal{M}}$ is

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