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On Normal τ-Measurable Operators Affiliated with Semifinite Von Neumann Algebras

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Abstract—Let τ be a faithful normal semifinite trace on the von Neumann algebra $\mathcal{M}$, $1 \geq q > 0$. The following generalizations of problems 163 and 139 from the book [1] to τ-measurable operators are obtained; it is established that: 1) each τ-compact q-hyponormal operator is normal; 2) if a τ-measurable operator A is normal and, for some natural number n, the operator A^n is τ-compact, then the operator A is also τ-compact. It is proved that if a τ-measurable operator A is hyponormal and the operator A^2 is τ-compact, then the operator A is also τ-compact. A new property of a nonincreasing rearrangement of the product of hyponormal and cohyponormal τ-measurable operators is established. For normal τ-measurable operators A and B, it is shown that the nonincreasing rearrangements of the operators AB and BA coincide. Applications of the results obtained to F-normed symmetric spaces on $(\mathcal{M}, \tau)$ are considered.

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INTRODUCTION

Let τ be a faithful normal semifinite trace on the von Neumann algebra $\mathcal{M}$, let $\widetilde{\mathcal{M}}$ be the $*$-algebra of all τ-measurable operators, and let $1 \geq q > 0$. In this paper, the following generalizations of problems 163 and 139 from the Halmos book [1] to τ-measurable operators are obtained; it is established that:

(1) each τ-compact q-hyponormal operator is normal (Theorem 2.2);

(2) if an operator $A \in \widetilde{\mathcal{M}}$ is normal and, for some natural number n, the operator A^n is τ-compact, then the operator A is τ-compact (item (i) of Corollary 3.2).

The proof of Theorem 2.2 is based on a deep result from [2]. It is shown by us that if the operator $A \in \widetilde{\mathcal{M}}$ is hyponormal and the operator A^2 is τ-compact, then the operator A is also τ-compact (item (i) of Corollary 3.4). We establish the new property of the nonincreasing rearrangement of the product of the hyponormal and cohyponormal τ-measurable operators (Theorem 3.5). For normal operators $A, B \in \widetilde{\mathcal{M}}$, it is shown that the nonincreasing rearrangements of the operators AB and BA coincide (Corollary 3.6). A well-known rearrangement property (see item (6) of Lemma 1.1) implies that a nonnegative operator $A \in \widetilde{\mathcal{M}}$ is τ-compact if and only if A^p is τ-compact for all $p > 0$. It is shown in Theorem 4.1 that a similar assertion also holds for the product of nonnegative operators $A, B \in \widetilde{\mathcal{M}}$: the τ-compactness of AB is equivalent to the τ-compactness of the operators A^pB^r for all $p, r > 0$. The results obtained are applied to F-normed symmetric spaces on $(\mathcal{M}, \tau)$.

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