Siberian Mathematical Journal, Vol. 55, *No.* 1, *pp.* 7–11, 2014

ON ADDITIVITY OF MAPPINGS ON MEASURABLE FUNCTIONS

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UDC 517.98

Abstract: We prove the additivity of regular l-additive mappings $T : \mathscr{K} \to [0, +\infty]$ of a hereditary cone $\mathscr{K}$ in the space of measurable functions on a measure space. Some examples are constructed of non-d-additive l-additive mappings T. The monotonicity of l-additive mappings $T : \mathscr{K} \to [0, +\infty]$ is established. The examples are constructed of nonmonotone d-additive mappings T.

Let $(S, +)$ be a commutative cancellation semigroup. Given a mapping $T : \mathscr{K} \to S$, we prove the equivalence of additivity and l-additivity. It is shown that a strongly regular 2-homogeneous l-subadditive mapping T is subadditive. All results are new even in case $\mathscr{K} = L_\infty^+$.

DOI: 10.1134/S0037446614010029

Keywords: measure space, measurable function, additive mapping, cone, weight, monotone mapping, cancellation semigroup, vector lattice

Introduction

Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space and let $\mathfrak{M} = \mathfrak{M}(\Omega, \mathfrak{A}, \mu)$ be the vector space (of the cosets) of measurable functions $f : \Omega \to \mathbb{R}$. Given $f, g \in \mathfrak{M}$, put $fg = 0$ if $\mu\{\omega \in \Omega : f(\omega)g(\omega) \neq 0\} = 0$.

Let $\mathscr{E}$ be a vector subspace of $\mathfrak{M}$. A functional $F : \mathscr{E} \to \mathbb{R}$ is called *disjointly additive* if from $f, g \in \mathscr{E}$ and $fg = 0$ it follows that $F(f + g) = F(f) + F(g)$. The integral representations for these functionals were obtained in [1–6] under extra conditions.

In integration theory, some important role is played by unbounded mappings $T : L_\infty^+ \to [0, +\infty]$. For a localizable measure space (see [7]) for normal homogeneous additive T and for normal monotone homogeneous subadditive T (see [8]), the representations were obtained via bounded linear functionals on L_∞.

Suppose that $\mathscr{K}$ is a hereditary cone in $\mathfrak{M}^+$; i.e., (1) $\lambda \in \mathbb{R}^+$, $f \in \mathscr{K} \Rightarrow \lambda f \in \mathscr{K}$; (2) $f, g \in \mathscr{K} \Rightarrow f + g \in \mathscr{K}$; and (3) $f \in \mathscr{K}$, $g \in \mathfrak{M}^+$ and $g \leq f \Rightarrow g \in \mathscr{K}$. Given $f, g \in \mathscr{K}$, define $f \vee g$ and $f \wedge g$ as

$$(f \vee g)(\omega) = \max\{f(\omega), g(\omega)\}, \quad (f \wedge g)(\omega) = \min\{f(\omega), g(\omega)\} \quad (\omega \in \Omega)$$

respectively. We have $f \vee g$, $f \wedge g \in \mathscr{K}$ and

$$f \vee g + f \wedge g = f + g. \tag{1}$$

A mapping $T : \mathscr{K} \to [0, +\infty]$ is called *l-additive* (i.e., lattice-additive) if

$$T(f \vee g) + T(f \wedge g) = T(f + g) \quad \text{for all } f, g \in \mathscr{K};$$

it is called *additive* if $T(f + g) = T(f) + T(g)$ for all $f, g \in \mathscr{K}$. By (1), every additive mapping is l-additive.

In this article we prove the additivity of regular l-additive mappings $T : \mathscr{K} \to [0, +\infty]$ (Theorem 2.1). We construct the examples of non-d-additive l-additive mappings T (Example 2.1). In Theorem 2.2, we establish the monotonicity of l-additive mappings $T : \mathscr{K} \to [0, +\infty]$. Example 2.1 shows the existence of nonmonotone d-additive mappings T.

Let $(S, +)$ be a commutative cancellation semigroup. We prove the equivalence of additivity and l-additivity for a mapping $T : \mathscr{K} \to S$ (Theorem 2.3).

Kazan. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 55, No. 1, pp. 11–16, January–February, 2014. Original article submitted January 30, 2013. Revision submitted October 18, 2013.