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International Journal of Theoretical Physics

ISSN 0020-7748
Volume 54
Number 6

Int J Theor Phys (2015) 54:1987-2000
DOI 10.1007/s10773-014-2405-1

Springer

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Received: 25 April 2014 / Accepted: 29 October 2014 / Published online: 18 November 2014

Abstract We introduce some new examples of quantum logics of idempotents in a ring. We continue the study of *symmetric logics*, i.e., collections of subsets generalizing Boolean algebras and closed under the symmetric difference.

Keywords Orthomodular poset · Quantum logic · State · Symmetric difference · Boolean algebra · Set representation · C^*-algebra · Von Neumann algebra · Positive functional · Trace · Idempotent · Projection · Additive mapping

1 Motivation

Orthomodular posets and, in particular, orthomodular lattices appear as algebraic structures of events in quantum mechanics, cf. [14, 17, 31, 32]. The natural requirement that the event system must allow "sufficiently many" states leads (in its stronger form) to orthomodular posets which can be represented as collections of subsets of a set generalizing σ-algebras [14]. In such collections, the set-theoretical symmetric difference can be introduced as an additional operation [29] which cannot be derived from the lattice-theoretical

Dedicated to memory of Professor Daniar Mushtari

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operations and orthocomplementation [21]. Thus we arrive at the notion of a symmetric logic.

During the study of symmetric logics, many questions remained open (cf. [5, 6]). In [7] we answered some of them. Here we present a generalization of [7, Theorem 4.3] with a shorter and direct proof.

2 Basic Notions

2.1 Quantum Logics of Idempotents of Unital Rings

Definition 2.1 Let $(L, \leq, 0, 1, ^{\perp})$ be a poset with 0 and 1 as the smallest and greatest element, respectively, and a unary operation $^{\perp} : L \to L$ (the *orthocomplementation*) such that

(i) $p \leq q \Rightarrow q^{\perp} \leq p^{\perp}, \quad p, q \in L$;
(ii) $(p^{\perp})^{\perp} = p, \quad p \in L$;
(iii) $p \vee p^{\perp} = 1, \quad p \in L$;
(iv) $p \leq q^{\perp} \Rightarrow p \vee q$ exists in $L, \quad p, q \in L$;
(v) $p \leq q \Rightarrow q = p \vee (p^{\perp} \wedge q), \quad p, q \in L.$

Then L will be called a *quantum logic* or also an orthomodular poset. If L is also a lattice, then L is called an *orthomodular lattice*.

Let $\mathcal{R}$ be a ring with unit e, $x^{\perp} := e - x$ for $\mathcal{R}$. Then $(x^{\perp})^{\perp} = x$. The set $\mathcal{R}^{\mathrm{id}} := \{x \in \mathcal{R} : \ x = x^2\}$, equipped with the partial order $p \leq q \Leftrightarrow pq = qp = p$ and orthocomplementation $p \mapsto p^{\perp}$, is a quantum logic. The logics $\mathcal{R}^{\mathrm{id}}$ are the main topic of this paper. They were investigated e.g. in [12, 13, 16, 18, 19, 25, 26].

Definition 2.2 Let $(L, \leq, 0, 1, ^{\perp})$ be a quantum logic. A subset S of L is said to be a sublogic of L if the following conditions are satisfied:

(i) $0 \in S$;
(ii) if $p \in S$ then $p^{\perp} \in S$;
(iii) if $p, q \in S$ and $p \leq q^{\perp}$, then $p \vee q \in S$.

Let $\mathcal{R}$ be an associative unital *-ring. Then the set $\mathcal{R}^{\mathrm{pr}} := \{x \in \mathcal{R} : \ x = x^* = x^2\}$ of all projections of $\mathcal{R}$ is a sublogic of the logic $\mathcal{R}^{\mathrm{id}}$. Let $\langle \mathcal{R}, \| \cdot \| \rangle$ be a unital Banach *-algebra, $\mathcal{R}_1 := \{x \in \mathcal{R} : \ \|x\| \leq 1\}$. A linear functional φ on $\mathcal{R}$ is called *positive* if $\varphi(x^*x) \geq 0$ for every $x \in \mathcal{R}$. Every positive linear functional φ on $\mathcal{R}$ is continuous and $\|\varphi\| = \varphi(e)$ [34, Chap. I, Lemma 9.9]. A positive linear functional of norm one is called a *state* [34, Chap. I, Definition 9.4].

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$, and $\mathcal{B}(\mathcal{H})$ be the *-algebra of all bounded linear operators on $\mathcal{H}$. The *strong (operator)* topology on $\mathcal{B}(\mathcal{H})$ is the locally convex topology determined by the seminorms $x \in \mathcal{B}(\mathcal{H}) \mapsto \|x\xi\|_{\mathcal{H}}$, $\xi \in \mathcal{H}$.

By the *commutant* of a set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ we mean the set

$$\mathcal{X}' = \{y \in \mathcal{B}(\mathcal{H}) : \ xy = yx, \ x^*y = yx^* \quad (x \in \mathcal{X})\}.$$

A *-subalgebra $\mathcal{R}$ of the algebra $\mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* acting in the Hilbert space $\mathcal{H}$ if $\mathcal{R} = \mathcal{R}''$. A complex Banach *-algebra $\mathcal{R}$ is called a C^**-algebra* if

$\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{R}$. Many C^*-algebras are generated as rings by their projections [1–4]. More precisely, every element in such a C^*-algebra $\mathcal{R}$ can be represented as a finite sum of finite products of projections from $\mathcal{R}$.

For C^*-algebra $\mathcal{R}$ let $\mathcal{R}^+$ denote its positive part. A linear functional $\varphi : \mathcal{R} \to \mathbb{C}$ is called a *trace* if $\varphi(z^*z) = \varphi(zz^*)$ for all $z \in \mathcal{R}$. A positive linear functional φ on a von Neumann algebra $\mathcal{R}$ is *normal* if $x_i \nearrow x \Longrightarrow \varphi(x) = \sup \varphi(x_i)$ $(x_i, x \in \mathcal{R}^+)$.

2.2 Concrete Logics

Let Ω be a non-empty set. By 2^Ω we denote the set of all subsets of Ω. For $n \in \mathbb{N}$, we define $\Omega_n = \{1, 2, \ldots, n\}$.

Let us recall [14] that a collection $\mathcal{E} \subseteq 2^\Omega$ of subsets of Ω is called a *concrete (quantum) logic* if the following conditions hold true:

(C1) $\Omega \in \mathcal{E}$,
(C2) $A \in \mathcal{E} \Rightarrow A^c := \Omega \setminus A \in \mathcal{E}$,
(C3) $A, B \in \mathcal{E},\ A \cap B = \varnothing \Rightarrow A \cup B \in \mathcal{E}$.

A concrete logic $\mathcal{E}$ is called a σ-class [14] if it satisfies the following strengthening of (C3):

(C3') $\{A_n \mid n \in \mathbb{N}\} \subseteq \mathcal{E},\ A_m \cap A_n = \varnothing$ whenever $m \neq n \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$.

A family $\mathcal{E} \subseteq 2^\Omega$ is a concrete logic if and only if it satisfies (C1) and the following condition:

(C4) $A, B \in \mathcal{E},\ A \subseteq B \Rightarrow B \setminus A \in \mathcal{E}$.

Remark 2.3 Every concrete logic can be represented as the logic of idempotents in some ring. Let Ω be a non-empty set, and let $\mathcal{E} \subseteq 2^\Omega$ be a concrete logic. If $\mathbb{R}^\Omega$ is the ring of all real functions on Ω, then the set of all characteristic functions χ_A, $A \in \mathcal{E}$, is a logic of idempotents of $\mathbb{R}^\Omega$. This logic is isomorphic to $\mathcal{E}$.

2.3 Symmetric Logics

The set 2^Ω is a group with respect to the symmetric difference operation: $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Notice that

$$A^c \triangle B = (A \triangle B)^c\,,$$
$$A^c \triangle B^c = A \triangle B\,.$$

A *symmetric logic* [28, Definition 3.2] is a concrete quantum logic $\mathcal{E}$ satisfying:

(S) $A, B \in \mathcal{E} \Rightarrow A \triangle B \in \mathcal{E}\,.$

A family $\mathcal{E} \subseteq 2^\Omega$ is a symmetric logic if and only if it satisfies (C1) and (S) [5, Proposition 1]. Symmetric logics were investigated e.g. in [5, 6, 10, 11, 21, 22, 28, 29].

Example 2.4 Let $n \in \mathbb{N}$ and $\Omega_{2n} = \{1, 2, \ldots, 2n\}$. Then the family

$$\mathcal{E}_{2n}^{\mathrm{even}} = \{A \subseteq \Omega_{2n} \mid \mathrm{card}\, A \text{ is even}\}$$

is a symmetric logic on Ω_{2n}.

Example 2.5 Let $\mathcal{E} \subset 2^\Omega$ be a concrete quantum logic and $T \in \mathcal{E}$, $T \neq \varnothing$. Then the family $\mathcal{E}_T = \{A \in \mathcal{E} \mid A \subseteq T\}$ is a concrete quantum logic with the greatest element T. Moreover, if $\mathcal{E}$ is a symmetric logic, then $\mathcal{E}_T$ is also a symmetric logic.

In the latter example, it was necessary to assume that $T \in \mathcal{E}$. This condition can be omitted in symmetric logics.

Example 2.6 Let $\mathcal{E} \subseteq 2^\Omega$ be a symmetric logic and $T \subseteq \Omega$, $T \neq \varnothing$. Then the family

$$\mathcal{E}|_T = \{A \cap T \mid A \in \mathcal{E}\} \subseteq 2^T$$

is a symmetric logic with the greatest element T.

2.4 States

We say that a mapping $m : \mathcal{E} \to [0, 1]$ is a *state* (or a finitely additive *probability measure*) on a concrete logic $\mathcal{E}$ if $m(\Omega) = 1$ and $m(A \cup B) = m(A) + m(B)$ whenever $A, B \in \mathcal{E}$, $A \cap B = \varnothing$. Let us denote by $P(\mathcal{E})$ the set of all states on a concrete logic $\mathcal{E}$. Recall that a state $m \in P(\mathcal{E})$ is called *subadditive* [31, p. 829] if for each $A, B \in \mathcal{E}$ there exists a set $C \in \mathcal{E}$ such that $C \supseteq A \cup B$ and $m(C) \leq m(A) + m(B)$.

If $\mathcal{E}$ is a Boolean algebra then any state $m \in P(\mathcal{E})$ is subadditive. There exists a concrete quantum logic which is not a Boolean algebra and all of its states are subadditive. This result was established in [30] with substantial help of the techniques developed in [23] and [27] (see also [31, p. 831]).

From now on, we suppose that $\mathcal{E}$ is a symmetric logic. A state $m \in P(\mathcal{E})$ is called Δ-subadditive [10] if

$$m(A \,\Delta\, B) \leq m(A) + m(B) \text{ for any pair } A, B \in \mathcal{E}.$$

The set of all Δ-subadditive states is convex. Every subadditive state $m \in P(\mathcal{E})$ is Δ-subadditive (hint: $C \supseteq A \cup B \supseteq A \,\Delta\, B$), but the reverse implication does not hold in general. In [6], the following situations were demonstrated:

1) a Δ-subadditive state which is not subadditive;
2) a two-valued state which is not Δ-subadditive.

3 Additive Mappings and Quantum Logics

3.1 New Quantum Logics of Idempotents in a Ring

Theorem 3.1 *Let $\mathcal{R}$ be a ring with unit e; $x, y \in \mathcal{R}$, and $\varphi : \mathcal{R} \to \mathbb{C}$ be an additive mapping with $\varphi(e) = 1$. Then the sets*

$$\mathcal{R}^{x,y}_{\varphi,1} := \{p \in \mathcal{R}^{\mathrm{id}} :\ \varphi(px + yp) = \varphi(p)\varphi(x + y)\}$$

and

$$\mathcal{R}^{x,y}_{\varphi,2} := \{p \in \mathcal{R}^{\mathrm{id}} :\ \varphi(xpy) = \varphi(p)\varphi(xy)\}$$

are quantum logics with the greatest element e, the partial order inherited from $\mathcal{R}^{\mathrm{id}}$ and the orthocomplementation $p \mapsto p^\perp = e - p$.

Moreover, if $\langle \mathcal{R}, t\rangle$ is a topological ring and φ is t-continuous, then the sets $\mathcal{R}^{x,y}_{\varphi,1}$ and $\mathcal{R}^{x,y}_{\varphi,2}$ are t-closed.

Proof It is clear that $0, e \in \mathcal{R}^{x,y}_{\varphi,k}$ for $k \in \{1,2\}$. We show that $p \in \mathcal{R}^{x,y}_{\varphi,k} \Leftrightarrow p^{\perp} \in \mathcal{R}^{x,y}_{\varphi,k}$ for all $p \in \mathcal{R}^{\mathrm{id}}$ and $k \in \{1,2\}$. Let $p \in \mathcal{R}^{x,y}_{\varphi,1}$. Since $p^{\perp}x + yp^{\perp} = x + y - (px + yp)$, we have

$$\varphi(p^{\perp}x + yp^{\perp}) = \varphi(x+y) - \varphi(px+yp) = \varphi(x+y) - \varphi(p)\varphi(x+y) = \varphi(p^{\perp})\varphi(x+y)$$

and $p^{\perp} \in \mathcal{R}^{x,y}_{\varphi,1}$. Let now $p \in \mathcal{R}^{x,y}_{\varphi,2}$. Since $xp^{\perp}y = xy - xpy$, we have

$$\varphi(xp^{\perp}y) = \varphi(xy) - \varphi(xpy) = \varphi(xy) - \varphi(p)\varphi(xy) = \varphi(p^{\perp})\varphi(xy)$$

and $p^{\perp} \in \mathcal{R}^{x,y}_{\varphi,2}$.

Let $p, q \in \mathcal{R}^{x,y}_{\varphi,k}$ for $k \in \{1,2\}$.

If $p \le q^{\perp}$, then $p \vee q = p + q \in \mathcal{R}^{\mathrm{id}}$ and it is easy to check that $p \vee q \in \mathcal{R}^{x,y}_{\varphi,k}$.

If $p \le q$, then $q - p \in \mathcal{R}^{\mathrm{id}}$, $q - p \le p^{\perp}$, and $q = (q-p) \vee p$. It is easy to check that $q - p \in \mathcal{R}^{x,y}_{\varphi,k}$.

Finally, note that if $\langle \mathcal{R}, t\rangle$ is a topological ring, then the quantum logic $\mathcal{R}^{\mathrm{id}}$, being defined by equalities containing continuous operations, is t-closed. □

Proposition 3.2 *Let $x, y, u, v \in \mathcal{R}$ and $p, q \in \mathcal{R}^{id}$. Then the following holds:*

1) $\mathcal{R}^{0,0}_{\varphi,1} = \mathcal{R}^{e,0}_{\varphi,1} = \mathcal{R}^{0,e}_{\varphi,1} = \mathcal{R}^{e,e}_{\varphi,1} = \mathcal{R}^{x,0}_{\varphi,2} = \mathcal{R}^{0,y}_{\varphi,2} = \mathcal{R}^{e,e}_{\varphi,2} = \mathcal{R}^{\mathrm{id}}$.
2) $\lambda, \mu \in \mathbb{Z} \Longrightarrow \mathcal{R}^{\lambda e \pm x, \mu e \pm y}_{\varphi,1} = \mathcal{R}^{x,y}_{\varphi,1}$ *for the following choices of signs in two $\pm$: $+, +$ and $-, -$.*
3) $\mathcal{R}^{-x,-y}_{\varphi,k} = \mathcal{R}^{x,y}_{\varphi,k}$ *for $k \in \{1,2\}$.*
4) $\mathcal{R}^{x,y}_{\varphi,1} \cap \mathcal{R}^{u,v}_{\varphi,1} \subset \mathcal{R}^{x+u,y+v}_{\varphi,1}$.
5) $\mathcal{R}^{x,0}_{\varphi,1} = \mathcal{R}^{e,x}_{\varphi,2}$.
6) $\mathcal{R}^{0,y}_{\varphi,1} = \mathcal{R}^{y,e}_{\varphi,2}$.
7) $p \in \mathcal{R}^{q,0}_{\varphi,1} \Leftrightarrow q \in \mathcal{R}^{0,p}_{\varphi,1}$.
8) $p \in \mathcal{R}^{q,q}_{\varphi,1} \Leftrightarrow q \in \mathcal{R}^{p,p}_{\varphi,1}$.
9) $p \in \mathcal{R}^{p,p}_{\varphi,1} \Leftrightarrow p \in \mathcal{R}^{p,p}_{\varphi,2} \Leftrightarrow \varphi(p) \in \{0,1\}$.

Proof 1) Easy verification.

2) We have

$$p(\lambda e \pm x) + (\mu e \pm y)p = (\lambda + \mu)p \pm (px + yp), \tag{1}$$

i.e. $\mp(px + yp) = (\lambda+\mu)p - (p(\lambda e \pm x) + (\mu e \pm y)p)$. The inclusion "$\subset$":

$$\begin{aligned}\mp\varphi(px+yp) &= (\lambda+\mu)\varphi(p) - \varphi(p(\lambda e \pm x) + (\mu e \pm y)p)\\ &= (\lambda+\mu)\varphi(p) - \varphi(p)\varphi((\lambda+\mu)e \pm (x+y))\\ &= (\lambda+\mu)\varphi(p) - \varphi(p)(\lambda+\mu \pm \varphi(x+y)) = \mp\varphi(p)\varphi(x+y).\end{aligned}$$

The inclusion "$\supset$": we have via (1)

$$\begin{aligned}\varphi(p(\lambda e \pm x) + (\mu e \pm y)p) &= \varphi((\lambda+\mu)p \pm (px+yp)) = (\lambda+\mu)\varphi(p) \pm \varphi(px+yp)\\ &= (\lambda+\mu)\varphi(p) \pm \varphi(p)\varphi(x+y)\\ &= \varphi(p)\varphi((\lambda e \pm x) + (\mu e \pm y)).\end{aligned}$$

3) For $k = 1$, it follows by 2) with $\lambda = \mu = 0$. For $k = 2$ we have $\varphi((-x)p(-y)) = \varphi(p)\varphi((-x)(-y)) \Leftrightarrow \varphi(xpy) = \varphi(p)\varphi(xy)$.
5) We have $\varphi(px) = \varphi(p)\varphi(x) \Leftrightarrow \varphi(epx) = \varphi(p)\varphi(ex)$.
6) We have $\varphi(yp) = \varphi(p)\varphi(y) \Leftrightarrow \varphi(ype) = \varphi(p)\varphi(ye)$.
7) We have $\varphi(pq + 0p) = \varphi(p)\varphi(q) \Leftrightarrow \varphi(q0 + pq) = \varphi(q)\varphi(p)$.
8) We have $\varphi(pq + qp) = \varphi(p)\varphi(2q) \Leftrightarrow \varphi(qp + pq) = \varphi(q)\varphi(2p)$.
9) We have $2\varphi(p) = \varphi(pp + pp) = \varphi(p)\varphi(p + p) \Leftrightarrow \varphi(p) = (\varphi(p))^2 \Leftrightarrow \varphi(p) \in \{0, 1\}$ and $\varphi(ppp) = \varphi(p)\varphi(pp) \Leftrightarrow \varphi(p) = (\varphi(p))^2 \Leftrightarrow \varphi(p) \in \{0, 1\}$.

□

Remark 3.3 We obtain $\mathcal{R}^{u,v}_{\varphi,1} \bigcap \mathcal{R}^{u+x,v+y}_{\varphi,1} \subset \mathcal{R}^{x,y}_{\varphi,1}$ by 3) and 4) of Proposition 3.2. If $\mathcal{R}$ is a unital algebra, then $\mathcal{R}^{\lambda e \pm x, \mu e \pm y}_{\varphi,1} = \mathcal{R}^{x,y}_{\varphi,1}$ for all $\lambda, \mu \in \mathbb{C}$ and for the following choices of signs in two $\pm$: $+, +$ and $-, -$.

Proposition 3.4 *Let $t \in \mathcal{R}$ be invertible, $\psi(z) := \varphi\left(tzt^{-1}\right)$ for all $z \in \mathcal{R}$ and let $p \in \mathcal{R}^{\mathrm{id}}$. Then $p \in \mathcal{R}^{x,y}_{\psi,k} \Leftrightarrow tpt^{-1} \in \mathcal{R}^{txt^{-1},tyt^{-1}}_{\varphi,k}$ for all $x, y \in \mathcal{R}$ and $k \in \{1, 2\}$.*

Proof The implication "⇒": If $k = 1$, then

$$\begin{aligned}\varphi\left(tpt^{-1}txt^{-1} + tyt^{-1}tpt^{-1}\right) &= \varphi\left(t(px + yp)t^{-1}\right) = \psi(px + yp) = \psi(p)\psi(x + y)\\ &= \varphi\left(tpt^{-1}\right)\varphi\left(txt^{-1} + tyt^{-1}\right).\end{aligned}$$

If $k = 2$, then

$$\begin{aligned}\varphi\left(txt^{-1}tpt^{-1}tyt^{-1}\right) &= \varphi\left(txpyt^{-1}\right) = \psi(xpy) = \psi(p)\psi(xy)\\ &= \varphi\left(tpt^{-1}\right)\varphi\left(txt^{-1}tyt^{-1}\right).\end{aligned}$$

□

Proposition 3.5 *Let $x, y \in \mathcal{R}$ and $p \in \mathcal{R}^{\mathrm{id}}$. If $py = yp$, then*

1) $p \in \mathcal{R}^{x,y}_{\varphi,1} \Leftrightarrow p \in \mathcal{R}^{x+y,0}_{\varphi,1}$;
2) $p \in \mathcal{R}^{x,y}_{\varphi,2} \Leftrightarrow p \in \mathcal{R}^{0,xy}_{\varphi,1}$.

In particular, if y is a central element of $\mathcal{R}$, then $\mathcal{R}^{x,y}_{\varphi,1} = \mathcal{R}^{x+y,0}_{\varphi,1}$ and $\mathcal{R}^{x,y}_{\varphi,2} = \mathcal{R}^{0,xy}_{\varphi,1}$.

3.2 Quantum Logics of Idempotents of Unital Banach *-algebras

Proposition 3.6 *Let $\langle \mathcal{R}, \|\cdot\| \rangle$ be a unital Banach *-algebra, $x, y \in \mathcal{R}$ and φ be a state on $\mathcal{R}$, $k \in \{1, 2\}$.*

1) *The quantum logic $\mathcal{R}^{x,y}_{\varphi,k}$ is $\|\cdot\|$-closed.*
2) $p \in \mathcal{R}^{x,y}_{\varphi,k} \Leftrightarrow p^* \in \mathcal{R}^{y^*,x^*}_{\varphi,k}$ *for all $p \in \mathcal{R}^{\mathrm{id}}$.*

Proof 1) The quantum logic $\mathcal{R}^{\mathrm{id}}$ is $\|\cdot\|$-closed. Every positive linear functional on any unital Banach *-algebra automatically is continuous [34, Chap. I, Lemma 9.9]. Hence the quantum logic $\mathcal{R}^{x,y}_{\varphi,k}$ is $\|\cdot\|$-closed via Theorem 3.1.

2) Recall that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. We have $\varphi(z^*) = \overline{\varphi(z)}$ for all $z \in \mathcal{R}$ [34, Chap. I, §9, formula (3)]. If $p \in \mathcal{R}_{\varphi,1}^{x,y}$, then

$$\varphi(p^*y^*+x^*p^*) = \varphi((px+yp)^*) = \overline{\varphi(px+yp)} = \overline{\varphi(p)}\cdot\overline{\varphi(x+y)} = \varphi(p^*)\varphi(x^*+y^*)$$

and $p^* \in \mathcal{R}_{\varphi,1}^{y^*,x^*}$. If $p \in \mathcal{R}_{\varphi,2}^{x,y}$, then

$$\varphi(y^*p^*x^*) = \varphi((xpy)^*) = \overline{\varphi(xpy)} = \overline{\varphi(p)} \cdot \overline{\varphi(xy)} = \varphi(p^*)\varphi(y^*x^*)$$

and $p^* \in \mathcal{R}_{\varphi,2}^{y^*,x^*}$.

In particular, for $y = x^*$ we have $p \in \mathcal{R}_{\varphi,k}^{x,x^*} \Leftrightarrow p^* \in \mathcal{R}_{\varphi,k}^{x,x^*}$ for all $p \in \mathcal{R}^{\mathrm{id}}$ and $k \in \{1, 2\}$. □

Theorem 3.7 *Let $\mathcal{R}$ be an unital C^*-algebra, $p \in \mathcal{R}^{\mathrm{id}}$ and $x \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) $xp = px$;
(ii) $p \in \mathcal{R}_{\varphi,1}^{x,e-x}$ *for all states φ on $\mathcal{R}$.*

Proof (ii)⇒(i). We have $\|\varphi\| = \varphi(e) = 1$ and $\varphi(xp) = \varphi(px)$ for all states φ on $\mathcal{R}$. By Hahn-Banach separation theorem, the set $\mathcal{R}^\star$ of all continuous linear functionals on $\mathcal{R}$ is separating for $\mathcal{R}$. If $f \in \mathcal{R}^\star$, we define $f^* \in \mathcal{R}^\star$ by setting $f^*(a) = \overline{f(a^*)}$ for all $a \in \mathcal{R}$. We say a functional $f \in \mathcal{R}^\star$ is *self-adjoint* if $f = f^*$. For any bounded linear functional f on $\mathcal{R}$, there are unique self-adjoint bounded linear functionals f_1 and f_2 on $\mathcal{R}$ such that $f = f_1 + \mathrm{i}f_2$ (take $f_1 = (f + f^*)/2$ and $f_2 = (f - f^*)/(2\mathrm{i})$). Let τ be a self-adjoint bounded linear functional on C^*-algebra $\mathcal{R}$. Then by Jordan Decomposition Theorem [24, Theorem 3.3.10] there exist positive linear functionals τ_+, τ_- on $\mathcal{R}$ such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$. Thus every $f \in \mathcal{R}^\star$ is a linear combination of four positive ones. Hence, the set of all states on $\mathcal{R}$ is separating for $\mathcal{R}$ and $xp = px$. □

Proposition 3.8 *Let a state φ on a von Neumann algebra $\mathcal{R}$ be normal, $x, y \in \mathcal{R}$ and $k \in \{1, 2\}$. Then the quantum logic $\mathcal{R}_{\varphi,k}^{x,y} \bigcap \mathcal{R}^{pr}$ is so-closed.*

Proof Since $\mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ is closed in the strong operator topology (i.e., *so*-closed) [15, Exercise 5.7.8] and $\mathcal{R}$ is *so*-closed, the set $\mathcal{R}^{\mathrm{pr}} = \mathcal{B}(\mathcal{H})^{\mathrm{pr}} \bigcap \mathcal{R}$ is *so*-closed. The multiplication operation $(u, v) \mapsto uv$ is *so*-continuous as a mapping $\mathcal{B}(\mathcal{H})_1 \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ [8, Chap. II, Proposition 2.4.1]. Finally, recall that every normal state φ on a von Neumann algebra $\mathcal{R}$ is *so*-continuous on $\mathcal{R}_1$ [34, Chap. II, Theorem 2.6]. □

Proposition 3.9 *If a state φ on a von Neumann algebra $\mathcal{R}$ is singular, then for every nonzero $p \in \mathcal{R}^{pr}$ there exists a nonzero $q \in \mathcal{R}^{pr}$ such that $q \le p$ and $q \in \mathcal{R}_{\varphi,1}^{p,0} \bigcap \mathcal{R}_{\varphi,1}^{0,p} \bigcap \mathcal{R}_{\varphi,1}^{p,p} \bigcap \mathcal{R}_{\varphi,2}^{p,p}$.*

Proof For singular state φ for every nonzero $p \in \mathcal{R}^{\mathrm{pr}}$ there exists a nonzero $q \in \mathcal{R}^{\mathrm{pr}}$ such that $q \le p$ and $\varphi(q) = 0$ [34, Chap. III, Theorem 3.8]. We have $pq = qp = \frac{1}{2}(pq + qp) = pqp = q$ and

$$\varphi(pq) = \varphi(qp) = \varphi(pq + qp) = \varphi(pqp) = \varphi(q) = 0 = \varphi(q)\varphi(p).$$ □

3.3 Quantum Logics and Tracial States on Unital C^*-algebras

Proposition 3.10 *Let φ be a tracial state on unital C^*-algebra $\mathcal{R}$ and $k \in \{1, 2\}$. Then the following holds:*

1) $\mathcal{R}^{x,y}_{\varphi,2} = \mathcal{R}^{yx,0}_{\varphi,1}$ *for all $x, y \in \mathcal{R}$.*
2) $\mathcal{R}^{x,y}_{\varphi,1} = \mathcal{R}^{x+y,e}_{\varphi,2}$ *for all $x, y \in \mathcal{R}$.*
3) $\mathcal{R}^{x,y}_{\varphi,2} = \mathcal{R}^{\mathrm{id}}$ *for all $x, y \in \mathcal{R}$ with $yx \in \{0, e\}$.*
4) $\mathcal{R}^{\lambda e \pm x, \mu e \mp x}_{\varphi,1} = \mathcal{R}^{\mathrm{id}}$ *for all $x \in \mathcal{R}$ and $\lambda, \mu \in \mathbb{C}$ (the signs in the formula must be opposite to each other).*
5) $\mathcal{R}^{x,x}_{\varphi,1} = \mathcal{R}^{x,x}_{\varphi,2}$ *for all $x \in \mathcal{R}^{\mathrm{id}}$.*
6) $\mathcal{R}^{x,x^\perp}_{\varphi,k} = \mathcal{R}^{\mathrm{id}}$ *for all $x \in \mathcal{R}^{\mathrm{id}}$.*
7) $p \in \mathcal{R}^{x,y}_{\varphi,k} \Leftrightarrow tpt^{-1} \in \mathcal{R}^{txt^{-1},tyt^{-1}}_{\varphi,k}$ *for all $p \in \mathcal{R}^{\mathrm{id}}$, $x, y \in \mathcal{R}$ and an invertible $t \in \mathcal{R}$.*

Proof 1) The inclusion "$\subset$": we have $\varphi(pyx) = \varphi(xpy) = \varphi(p)\varphi(xy) = \varphi(p)\varphi(yx)$. The inclusion "$\supset$": we have $\varphi(xpy) = \varphi(pyx) = \varphi(p)\varphi(yx) = \varphi(p)\varphi(xy)$.

2) The inclusion "$\subset$": we have $\varphi(p)\varphi(x+y) = \varphi(px+yp) = \varphi(px)+\varphi(yp) = \varphi((x+y)p) = \varphi((x+y)pe)$. The inclusion "$\supset$": we have $\varphi(px+yp) = \varphi(px)+\varphi(yp) = \varphi(xp)+\varphi(yp) = \varphi((x+y)p) = \varphi((x+y)pe) = \varphi(p)\varphi(x+y)$.

3) Let $p \in \mathcal{R}^{\mathrm{id}}$. If $yx = 0$, then $0 = \varphi(pyx) = \varphi(xpy) = \varphi(p)\varphi(xy) = \varphi(p)\varphi(yx)$. If $yx = e$, then $\varphi(xpy) = \varphi(pyx) = \varphi(p) = \varphi(p)\varphi(yx) = \varphi(p)\varphi(xy)$.

4) We have

$$\begin{aligned}\varphi(p(\lambda e \pm x) + (\mu e \mp x)p) &= \varphi((\lambda+\mu)p \pm (px - xp)) \\ &= (\lambda+\mu)\varphi(p) \pm (\varphi(px) - \varphi(xp)) \\ &= (\lambda+\mu)\varphi(p) = \varphi(p)\varphi((\lambda e \pm x) + (\mu e \mp x)))\end{aligned}$$

for all $p \in \mathcal{R}^{\mathrm{id}}$.

5) The inclusion "$\subset$": we have $\varphi(px+xp) = \varphi(px)+\varphi(xp) = 2\varphi(px) = \varphi(p)\varphi(2x) \Rightarrow \varphi(xpx) = \varphi\left(px^2\right) = \varphi(px) = \varphi(p)\varphi\left(x^2\right)$.

The inclusion "$\supset$": we have $\varphi(xpx) = \varphi\left(px^2\right) = \varphi(p)\varphi\left(x^2\right) = \varphi(p)\varphi(x) \Rightarrow$

$$\begin{aligned}\Rightarrow \varphi(px+xp) &= \varphi(px)+\varphi(xp) = 2\varphi(xpx) = 2\varphi(p)\varphi(x^2) = 2\varphi(p)\varphi(x) \\ &= \varphi(p)\varphi(x+x).\end{aligned}$$

6) Let $p \in \mathcal{R}^{\mathrm{id}}$. If $k = 1$, then

$$\varphi(px + x^\perp p) = \varphi(px) + \varphi(x^\perp p) = \varphi(px + px^\perp) = \varphi(p) = \varphi(p)\varphi(x + x^\perp).$$

If $k = 2$, then $\varphi\left(xpx^\perp\right) = \varphi\left(px^\perp x\right) = \varphi(0) = 0 = \varphi(p)\varphi\left(xx^\perp\right)$.

7) We apply Proposition 3.4 with $\psi = \varphi$. □

Example 3.11 Let $\mathcal{R} = \mathbb{M}_2(\mathbb{C})$ and φ be the normalized trace on $\mathcal{R}$, i.e. $\varphi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\right) = \frac{1}{2}(\alpha+\delta)$, $0 = \mathrm{diag}(0,0)$, $e = \mathrm{diag}(1,1)$. Put $p(a,b,c) = \begin{pmatrix} a & b \\ c & 1-a\end{pmatrix}$ for $a, b, c \in \mathbb{C}$ with $a = a^2 + bc$, then

$$\mathcal{R}^{\mathrm{id}} = \{0,\ e,\ p(a,b,c) \text{ with } a,b,c \in \mathbb{C} \text{ and } a = a^2 + bc\}$$

is a quantum logic which is a lattice. For $x = p(1, 0, 0)$ and $y = p(1/2, 1/2, 1/2)$ we have

$$\mathcal{R}^{x,y}_{\varphi,1} = \{0, e,\ p(a, b, c),\ \text{ where } a, b, c \in \mathbb{C} \text{ with } a = a^2 + bc \text{ and } 2a + b + c = 1\},$$

$$\mathcal{R}^{x,y}_{\varphi,2} = \{0, e,\ p(a, b, c),\ \text{ where } a, b, c \in \mathbb{C} \text{ with } a = a^2 + bc \text{ and } 2a + 2b = 1\}.$$

Hence $\mathcal{R}^{x,y}_{\varphi,1} \cap \mathcal{R}^{x,y}_{\varphi,2} = \left\{0, e, q = p\left(\frac{1}{2} - \frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}\right), q^{\perp}\right\}$. Also we have

$$p(0, 1, 0) \in \mathcal{R}^{x,y}_{\varphi,1} \setminus \mathcal{R}^{x,y}_{\varphi,2}, \quad p(1/4, 1/4, 3/4) \in \mathcal{R}^{x,y}_{\varphi,2} \setminus \mathcal{R}^{x,y}_{\varphi,1}.$$

4 Concrete Quantum Logics

4.1 Asymmetric Logics: Definition and Examples

Definition 4.1 A concrete logic $\mathcal{E}$ is called an *asymmetric logic* if $A \bigtriangleup B \in \mathcal{E}$ if and only if $A \cap B \in \mathcal{E}$ for all $A, B \in \mathcal{E}$.

Example 4.2 Let $\Omega = \{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\Omega \in \ell_1$, i.e. the series $\sum_{n=1}^{\infty} z_n$ converges absolutely. Let $\Lambda \in \{\mathbb{Q}, \mathbb{R}\}$ and $z = \sum_{n=1}^{\infty} z_n$. Recall that every rearrangement of $\{z_n\}_{n=1}^{\infty}$ preserves the absolute convergence and the sum z. Then

$$\mathcal{E}_{\Lambda,\Omega} = \{I \subset \Omega \mid \sum_{x \in I} x = \lambda z \quad \text{for some} \quad \lambda \in \Lambda\}$$

is an asymmetric logic. (The sum of an empty sequence is considered zero, thus $\varnothing \in \mathcal{E}_{\Lambda,\Omega}$.) Moreover, $\mathcal{E}_{\mathbb{R},\Omega}$ is a σ-class and $\mathcal{E}_{\mathbb{Q},\Omega}$ is its sublogic.

Example 4.3 Let $\mathcal{A}$ be the Lebesgue σ-algebra on $\Omega = [0, 1]$, μ be the linear Lebesgue measure such that $\mu(\Omega) = 1$. Then $\mathcal{E}_{\mathbb{Q},\mu} = \{A \in \mathcal{A} :\ \mu(A) \in \mathbb{Q}\}$ is an asymmetric logic.

Symmetric logics may be assymetric, e.g., Boolean algebras, or may not be assymetric, e.g. $\mathcal{E}_4^{\text{even}}$. The latter example is prototypical in the following sense:

Proposition 4.4 *If $\mathcal{E}$ is a symmetric logic of subsets of Ω and $\mathcal{E}$ is not an asymmetric logic, then there is a partition $\{C_i\}_{i=1}^{4}$ of Ω with the following property:*

For $I \subset \{1, 2, 3, 4\}$, the union $\bigcup_{i \in I} C_i$ belongs to $\mathcal{E}$ if and only if card I *is even.*

Proof If $\mathcal{E}$ is not an asymmetric logic, then there are $A, B \in \mathcal{E}$ such that $A \bigtriangleup B \in \mathcal{E}$ and $A \cap B \notin \mathcal{E}$. It suffices to take $C_1 = A \cap B^c$, $C_2 = A^c \cap B$, $C_3 = A \cap B$, $C_4 = A^c \cap B^c$. □

Proposition 4.5 *A symmetric logic is an asymmetric logic if and only if it is a Boolean algebra.*

Together with Proposition 4.4, we obtain:

Corollary 4.6 *If a symmetric logic is not a Boolean algebra, it contains a sublogic isomorphic to $\mathcal{E}_4^{even}$.*

4.2 Concrete Logics Generated by the Independence Relation

Let $\mathcal{A}$ be a Boolean algebra with the unit Ω, $\varphi : \mathcal{A} \to \mathbb{C}$ be an additive mapping ($\varphi(A \cup B) = \varphi(A) + \varphi(B)$ for all $A, B \in \mathcal{A}$, $A \cap B = \varnothing$) with $\varphi(\Omega) = 1$. Let $A, B \in \mathcal{A}$. We have $\varphi(A) + \varphi(A^c) = \varphi(\Omega) = 1$ and $\varphi(A^c) = 1 - \varphi(A)$, hence $\varphi(\varnothing) = 0$. The following conditions are equivalent:

(i) $\varphi(A \cap B) = \varphi(A)\varphi(B)$;
(ii) $\varphi(A^c \cap B) = \varphi(A^c)\varphi(B)$;
(iii) $\varphi(A \cap B^c) = \varphi(A)\varphi(B^c)$;
(iv) $\varphi(A^c \cap B^c) = \varphi(A^c)\varphi(B^c)$.

Proposition 4.7 *The family*

$$\mathcal{A}_\varphi^A := \{B \in \mathcal{A} : \ \varphi(A \cap B) = \varphi(A)\varphi(B)\}$$

is a concrete logic with the greatest element Ω. We have $\mathcal{A}_\varphi^A = \mathcal{A}_\varphi^{A^c}$. Moreover, if $\mathcal{A}$ is a σ-algebra and φ is σ-additive, then $\mathcal{A}_\varphi^A$ is a σ-class.

Proof It follows by distributivity of the intersection with respect to the union. □

Let $\mathcal{A}$ be a Boolean algebra and $\nu : \mathcal{A} \to \mathbb{R}$ be a measure ($\nu(A \cup B) = \nu(A) + \nu(B)$ for all $A, B \in \mathcal{A}$, $A \cap B = \varnothing$). An event $A \in \mathcal{A}$ is a ν-atom if $\nu(A) > 0$ and if for any event $B \subset A$, either $\nu(B) = \nu(A)$ or $\nu(B) = 0$. A measure ν is *nonatomic* if it has no ν-atoms. A state ν is *purely atomic*, if there is a sequence of ν-atoms such that the sum of their probabilities is 1.

Remark 4.8 We have $\mathcal{A}_\varphi^\varnothing = \mathcal{A}_\varphi^\Omega = \mathcal{A}$ and $A \in \mathcal{A}_\varphi^A \Leftrightarrow \varphi(A) \in \{0, 1\}$. Moreover, if $\varphi : \mathcal{A} \to [0, 1]$, then $\mathcal{A}_\varphi^A = \mathcal{A}$ for all $A \in \mathcal{A}$ with $\varphi(A) \in \{0, 1\}$. If φ is nonatomic, then there exists nonempty $A \in \mathcal{A}$ with $\varphi(A) = 0$ [20].

Theorem 4.9 *$\mathcal{A}_\varphi^A$ is an asymmetric logic.*

Proof We show that for $B, C \in \mathcal{A}_\varphi^A$ the following conditions are equivalent:

(i) $B \triangle C \in \mathcal{A}_\varphi^A$;
(ii) $B \cap C \in \mathcal{A}_\varphi^A$.

Recall that $\varphi(A \cap B) = \varphi(A)\varphi(B)$ and $\varphi(A \cap C) = \varphi(A)\varphi(C)$. The implication (i)⇒(ii): we have

$$\varphi(A \cap (B \triangle C)) = \varphi(A)\varphi(B \triangle C) = \varphi(A)(\varphi(B) + \varphi(C) - 2\varphi(B \cap C)) \tag{2}$$

and via distributivity of the intersection with respect to the symmetric difference

$$\begin{aligned} \varphi(A \cap (B \triangle C)) &= \varphi((A \cap B) \triangle (A \cap C)) \\ &= \varphi(A \cap B) + \varphi(A \cap C) - 2\varphi(A \cap B \cap C) \\ &= \varphi(A)\varphi(B) + \varphi(A)\varphi(C) - 2\varphi(A \cap B \cap C). \end{aligned}$$

Now via (2) we obtain $\varphi(A \cap (B \cap C)) = \varphi(A)\varphi(B \cap C)$, as desired.

The implication (ii)⇒(i) can be verified by inversion of the chain of arguments given above. □

Corollary 4.10 *If a concrete logic $\mathcal{A}_\varphi^A$ is a symmetric logic, then it is a Boolean algebra.*

Corollary 4.11 *For $n \geq 2$ the symmetric logic $\mathcal{E}_{2n}^{even}$ cannot be represented in the form $\mathcal{A}_\varphi^A$ with some $\mathcal{A}$, φ and $A \in \mathcal{A}$.*

Proposition 4.12 *Let $\mathcal{A}$ be a Boolean algebra and $\varphi, \psi \in P(\mathcal{A})$ be so that at least one of them is nonatomic. If $\mathcal{A}_\varphi^A = \mathcal{A}_\psi^A$ for all $A \in \mathcal{A}$, then $\varphi = \psi$.*

Proof Note that φ, ψ have identical independent events (i.e. for any pair of events A and B, $\varphi(A \cap B) = \varphi(A)\varphi(B)$ if and only if $\psi(A \cap B) = \psi(A)\psi(B)$) and apply Theorem 1 of [9]. □

Example 4.13 Let $\mathcal{A} = 2^{\Omega_6}$, $\varphi(X) = \frac{1}{6}\text{card}\, X$ for $X \in \mathcal{A}$. Let $A = \{2, 4, 6\}$. Then

$$\mathcal{A}_\varphi^A = \{\varnothing, \Omega_6, B = \{1,2\}, C = \{1,4\}, D = \{1,6\}, E = \{2,3\}, F = \{2,5\},$$
$$G = \{3,4\}, H = \{3,6\}, I = \{4,5\}, J = \{5,6\}, B^c, C^c, D^c, E^c, F^c, G^c, H^c, I^c, J^c\}.$$

We have $B^c \Delta H = I$ and $B \Delta C \notin \mathcal{A}_\varphi^A \subset \mathcal{E}_6^{\text{even}}$.

Example 4.14 Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathcal{A} = 2^{\mathbb{N}_0}$ and a state φ be defined by a non-increasing sequence $a_n = \varphi(\{n\})$, $n \in \mathbb{N}_0$. If $a_{n+1} \leq a_n^2$ holds for all $n \in \mathbb{N}_0$, then there are no (nontrivial) independent events in this probability space [33, Example 1.1]. Thus $\mathcal{A}_\varphi^A = \{\varnothing, \mathbb{N}_0\}$ for all $A \in \mathcal{A} \setminus \{\varnothing, \mathbb{N}_0\}$.

Remark 4.15 The range of a purely atomic probability measure can easily be the whole $[0, 1]$, e.g. if the probability of the n-th atom is $a_n = 1/2^{n+1}$. If the range $\{\varphi(A) : A \in \mathcal{A}\}$ of a probability measure φ contains the whole interval $[0, 1]$ or at least if the range contains an arbitrary small interval $[0, \varepsilon]$, $\varepsilon > 0$, then there are infinitely many independent events in the underlying probability space [33, Theorem 1.1].

4.3 When All States are Δ-subadditive

All states on Boolean algebras are subadditive and hence Δ-subadditive.

Problem 4.16 [6, Problem 7.1] Let $\mathcal{E}$ be a symmetric logic such that any state $m \in P(\mathcal{E})$ is Δ-subadditive. Is it true that $\mathcal{E}$ is a Boolean algebra?

A positive answer was given in [7, Theorem 4.3] with a proof by induction on the cardinality of the domain. Here we present a more general result with a new proof which is shorter and constructive—we describe the state which violates Δ-subadditivity.

Let us recall that a state m_x on a concrete logic $\mathcal{E}$ of subsets of Ω is *concentrated* in a point $x \in \Omega$ if

$$m_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.17 *Let $\mathcal{E}$ be a finite symmetric logic with the following property:*

Each state on $\mathcal{E}$ which is an affine combination of concentrated states is Δ-subadditive.

Then $\mathcal{E}$ is a Boolean algebra.

Proof Suppose that $\mathcal{E}$ is a finite symmetric logic of subsets of Ω. Without loss of generality, we assume that $\mathcal{E}$ satisfies

$$\forall a, b \in \Omega, a \neq b\, \exists A \in \mathcal{E} : a \in A, b \notin A\,.$$

This means that each two points $a, b \in \Omega$ can be separated by an element of $\mathcal{E}$. Such a representation can be always found by the identification of points which cannot be separated. As $\mathcal{E}$ is finite, so is Ω. Let $n = \operatorname{card}\Omega$.

For $x \in \Omega$, we define

$$\mathcal{E}_x = \{A \in \mathcal{E} \mid x \in A\}\,.$$

According to our assumptions, $\bigcap \mathcal{E}_x = \{x\}$ for all $x \in \Omega$.

If $\mathcal{E}$ contains all singletons, it is a Boolean algebra isomorphic to 2^Ω. Suppose that $\{x\} \notin \mathcal{E}$. We choose two sets $A, B \in \mathcal{E}_x$ such that their intersection, $A \cap B$, has the least possible cardinality, say k.

Claim $A \cap B$ is a proper subset of A and B, i.e., there exist $a \in A \setminus B$, $b \in B \setminus A$.

Proof of the claim If $A \subset B$ and $\operatorname{card} A > 1$, then there is a $c \in A$, $c \neq x$. As c can be separated from x, there is a $C \in \mathcal{E}$ such that $x \in C$, $c \notin C$. The intersection $A \cap C$ contains x and has a lower cardinality than $A = A \cap B$, a contradiction.

As a corollary, we get the following:

Claim Each set from $\mathcal{E}$ has at least $k + 1$ elements.

Now we are ready to finish the proof of the theorem. We define m as the following affine combination of concentrated states:

$$m = \frac{-k}{n-k-1}\, m_x + \frac{1}{n-k-1} \sum_{y \neq x} m_y\,,$$

where the sum is taken over all $y \in \Omega \setminus \{x\}$. Due to the preceding claim, m is non-negative. As an affine combination of states, m is additive and satisfies $m(\Omega) = 1$, thus it is a state. However, m is not Δ-subadditive because

$$\begin{aligned}
m(A) &= \frac{1}{n-k-1}\,(-k + \operatorname{card} A - 1)\,,\\
m(B) &= \frac{1}{n-k-1}\,(-k + \operatorname{card} B - 1)\,,\\
m(A) + m(B) &= \frac{1}{n-k-1}\,(-2\,k + \operatorname{card} A + \operatorname{card} B - 2)\,,\\
m(A \Delta B) &= \frac{1}{n-k-1}\,(-2\,k + \operatorname{card} A + \operatorname{card} B) > m(A) + m(B)\,.
\end{aligned}$$

□

Remark 4.18 Theorem 4.17 cannot be extended to infinite symmetric logics, see Proposition 4.8 of [7].

Acknowledgments The second author was supported by the Czech Technical University in Prague under project SGS12/187/OHK3/3T/13.

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